Sundials and Linear Algebra

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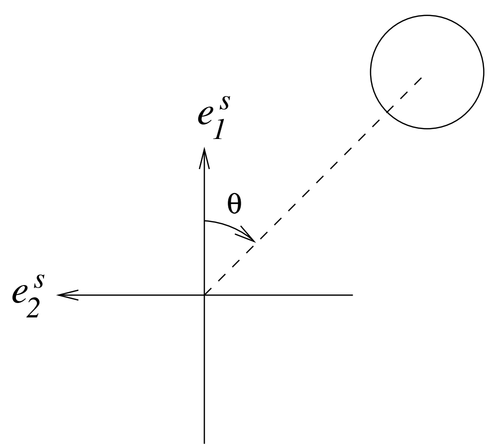
Most texts on creating sundials are directed towards those who are solely interested in making and using sundials and usually assumes minimal mathematical background. Very little, if any, effort is put into describing the mathematical background of why the equations work or where they come from. This document is to help those students that have a little linear algebra knowledge see one method for deriving those common equations.

Laying the Foundation: Pre-Copernican Assumptions

To start the derivation we will assume that we are located at the center of the solar system and that the sun orbits us at a constant angular velocity and completes one full cycle every 24 hours.

Let's define our right-handed orthonormal basis vectors e_1^s, e_2^s, and e_3^s as the solar system coordinate basis and also define θ such that the the sun's position is measured clockwise from e_1^s.



In this image, e_3^s is going out of the page. This is set up as though we are looking down on the north pole of a non-rotating earth with e_2^s pointing west (pick a longitude) and e_3^s pointing up towards Polaris. For this setup, we assume that at midnight the sun is in the e_1^s direction ($\theta = 0$) and at noon the sun is at $\theta = \pi$, and so on. So, for a 24 hour day we define θ by

$$\theta = 2\pi \frac{h}{24} \tag{1}$$

where h is the local solar time (the time that you would be using if not for daylight savings time and timezones). By definition, $h = 12$ at noon and $h = 0$ at midnight. This definition makes it easy to calculate the (e_1^s, e_2^s) direction of the sun at any given time (making sure to take into account our particular definition of θ)

$$s^s = \begin{Bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{Bmatrix} \tag{2}$$

where s^s is the vector pointing to the sun in the solar system coordinate basis.

The vector in Equation 2 assumes that the sun's declination is zero which is true during an equinox, but is not true generally. The sun's declination δ is a minimum at the winter solstice ($\delta = -23.45°$) and maximum at the summer solstice ($\delta = 23.45°$). Taking into account the declination of the sun, we have a polar coordinate system in θ and δ. We can write s^s as a unit vector in standard cartesian-from-polar form

$$s^s = \begin{Bmatrix} \cos\delta\cos\theta \\ -\cos\delta\sin\theta \\ \sin\delta \end{Bmatrix} \tag{3}$$

Notice that we define it in such a way so that a positive declination (in the summer) gives a positive e_3^s displacement. Be sure to give declination in radians if your trigonometric functions require it.

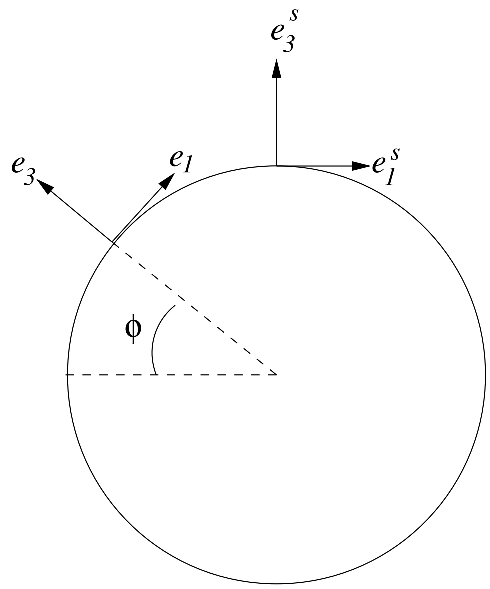
For convenience, here is a very approximate equation for solar declination (in degrees) for a given date

$$\delta = -23.44° \cos\left(360° \frac{N + 10}{365}\right) \tag{4}$$

where N is the day of the year. The maximum error is approximately 1 degree.

Rotating to Account for Observer Latitude

We now need to take into account the fact that we are located on the surface of a celestial body that can be approximated as a sphere. Because we are found at different latitudes we need to perform a basis transform on the s^s vector given in Equation 3 to give us a vector in our own basis for our given latitude (represented by ϕ).



In this image, e_2 and e_2^s are pointing into the page. From this image we can readily see that we need to rotate s^s about e_2^s (which is equal to e_2). The general rotation tensor for a rotation about e_2 of angle α is given by

$$R = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{5}$$

For a person located at the equator we would need to rotate by $\alpha = \pi/2$. For a person at a given latitude ϕ the rotation is $\alpha = \pi/2 - \phi$. If we plug this definition of α into Equation 5 and simplify, we get

$$R = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \\ -\cos\phi & 0 & \sin\phi \end{bmatrix} \tag{6}$$

Then, we can get our new, rotated s^s vector by

$$s = R \cdot s^s = \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \\ -\cos\phi & 0 & \sin\phi \end{bmatrix} \cdot \begin{Bmatrix} \cos\delta\cos\theta \\ -\cos\delta\sin\theta \\ \sin\delta \end{Bmatrix} \tag{7}$$

The vector s can be written out (only using basic terms) as

$$s = \begin{Bmatrix} \sin\delta\cos\phi + \cos\delta\sin\phi\cos\left(2\pi\frac{h}{24}\right) \\ -\cos\delta\sin\left(2\pi\frac{h}{24}\right) \\ \sin\delta\sin\phi - \cos\delta\cos\phi\cos\left(2\pi\frac{h}{24}\right) \end{Bmatrix} \tag{8}$$

which still retains the unit-vector property because we simply rotated a unit vector. This is our heliostat vector that always points towards the sun, even at night

Altitude and Azimuth

We can now calculate the altitude and azimuth of the sun for a given hour.

$$\theta_{\text{alt}} = \sin^{-1}(s \cdot e_3) \tag{9}$$

$$\theta_{\text{az}} = 2\pi - \tan^{-1}\left(\frac{s \cdot e_2}{s \cdot e_1}\right) \tag{10}$$

The factor of 2π in the definition of the azimuthal angle is to account for the fact that the arctangent will give positive angle values for counterclockwise angles while azimuth is generally measured in the clockwise direction from north (e_1).

Sunrise and Sunset

We can compute when the sun will rise and set (or find out if the sun will rise or set at all during a given day) by looking at Equation 8 and recognizing that the sun is up when $s \cdot e_3$ is positive and negative when the sun is down.

To find when the sun rises or sets, we set $s \cdot e_3 = 0$ and solve for h:

$$\sin\delta \sin\phi - \cos\delta \cos\phi \cos\left(2\pi \frac{h}{24}\right) = 0 \tag{11}$$

$$\tan\delta \tan\phi = \cos\left(2\pi \frac{h}{24}\right) \tag{12}$$

$$h = \frac{24}{2\pi} \cos^{-1}(\tan\delta \tan\phi) \tag{13}$$

Because of the arc-cosine we have (at most) two possible solutions on the domain of $0 \leq h \leq 24$ (one full day). The two solutions are given by

$$h_{rise} = \frac{24}{2\pi} \cos^{-1}(\tan\delta \tan\phi) \tag{14}$$

$$h_{set} = -\frac{24}{2\pi} \cos^{-1}(\tan\delta \tan\phi) = 24 - \frac{24}{2\pi} \cos^{-1}(\tan\delta \tan\phi) \tag{15}$$

When $\tan\delta \tan\phi < -1$ the sun never rises and when it is greater than one the sun never sets.

Casting Shadows

If we want to determine the length and direction of a shadow cast by an object we need only define a vector representing the object (such as a person standing or the gnomon of a sundial) and the normal to the surface onto which we want to project the shadow. For example, if we want to know about the shadow of a person that is 180cm tall we define the object vector x as

$$x = \begin{Bmatrix} 0 \\ 0 \\ 180 \end{Bmatrix} \tag{16}$$

and the normal to the plane as $n = e_3$. Then, the shadow vector q is given by

$$q = x - \frac{s(n \cdot x)}{s \cdot n} \tag{17}$$

Example 1

As an example, if a person that lives at $\phi = 42°$ (Chicago, IL) and is 180cm tall goes outside on the summer solstice ($\delta = 23.45°$) at 2:00pm ($h = 14$, local solar time) the shadow will extend 49.5cm to the North and 96.3cm to the East. The closed-form solution for an object of unit height is

$$q = \frac{1}{\sin\delta \sin\phi - \cos\delta\cos\phi\cos\left(2\pi\frac{h}{24}\right)} \begin{Bmatrix} \sin\delta\cos\phi + \cos\delta\sin\phi\cos\left(2\pi\frac{h}{24}\right) \\ \cos\delta\sin\left(2\pi\frac{h}{24}\right) \\ 0 \end{Bmatrix} \tag{18}$$

Example 2

If we want to determine the hour lines for a classical horizontal dial we need to determine the angle between the shadow of the gnomon and the noon mark. For this we need to define

$$x = \begin{Bmatrix} \cos\phi \\ 0 \\ \sin\phi \end{Bmatrix} \tag{19}$$

$$n = e_3 \tag{20}$$

For this special case where the gnomon is in line with the axis of rotation of the earth and we are only interested in the direction (not length) of the shadow, we can assume that $\delta = 0$. Then the equation of q simplifies to

$$q = \begin{Bmatrix} \frac{1}{\cos\phi} \\ -\tan\phi\tan\left(2\pi\frac{h}{24}\right) \\ 0 \end{Bmatrix} \tag{21}$$

The angle γ between the shadow and the noon mark can be found by

$$\tan\gamma = \frac{-\tan\phi\tan\left(2\pi\frac{h}{24}\right)}{\frac{1}{\cos\phi}} \tag{22}$$

$$\tan\gamma = -\sin\phi\tan\left(2\pi\frac{h}{24}\right) \tag{23}$$

which is the same equation that is given in elementary sundial books (the negative sign can be ignored if all you want is the magnitude of the angle).