$$\frac{ds}{dt} = \frac{\partial s}{\partial A_{11}}\frac{dA_{11}}{dt} + \frac{\partial s}{\partial A_{12}}\frac{dA_{12}}{dt} + \ldots + \frac{\partial s}{\partial A_{33}}\frac{dA_{33}}{dt}. \tag{17.16}$$

Letting $\frac{ds}{d\underset{\approx}{A}}$ denote the tensor whose components are $\frac{\partial s}{\partial A_{ij}}$, this result is written

$$\frac{ds}{dt} = \frac{ds}{d\underset{\approx}{A}} : \frac{d\underset{\approx}{A}}{dt}. \tag{17.17}$$

The "$:$" operator, not "$\bullet\bullet$", appears naturally in the chain rule; each component of one tensor is multiplied by the *corresponding* component of the other tensor.

Third and Fourth-order tensor inner product

The **inner product** between two third-order tensors, $\underset{\approx}{X}$ and $\underset{\approx}{Y}$, is a scalar given by $X_{ijk}Y_{ijk}$, which is an implied summation of 27 terms that multiply each component of $\underset{\approx}{X}$ times the corresponding component of $\underset{\approx}{Y}$.

The inner product between fourth-order tensors, $\underset{\approx}{X}$ and $\underset{\approx}{Y}$, is a scalar denoted $\underset{\approx}{X}::\underset{\approx}{Y}$ and defined

$$\underset{\approx}{X}::\underset{\approx}{Y} \equiv X_{ijkl}Y_{ijkl}. \tag{17.18}$$

This implied summation consists of a whopping 81 terms over every component of $\underset{\approx}{X}$ multiplied by the corresponding component of $\underset{\approx}{Y}$.

It is no coincidence that the inner products for third- and higher-order tensors are analogous to those for vectors and second-order tensors. By applying a mathematician's definition of a vector, the set of all fourth-order tensors can be shown to be an abstract 81-dimensional vector space. Although this view is occasionally useful in applications, we will usually find that fourth-order tensors are most conveniently regarded as operations (such as material constitutive laws) that transform second-order tensors to second-order tensors. Hence, fourth-order tensors referenced to a 3D space (which we term as type V_3^4) may be regarded as *second-order* tensors referenced to nine-dimensional tensor space (type V_9^2). As discussed on page 351, this rigorously justifies writing the components of fourth-order tensors as 9×9 matrices. Minor-symmetric tensors may be interpreted as being of type V_6^2, allowing 6×6 component matrices, as is common in constitutive modeling (see "Voigt" and "Mandel" in the index).

Fourth-order Sherman-Morrison formula

When regarding second-order tensors as nine-dimensional vectors, the inner product is the tensor inner product (i.e., the double-dot product). Formulas for ordinary 3D vectors have generalizations to this higher-dimensional space. This notion was first mentioned in Eqs. (17.4) and (17.6). As another example, we note that a rank-one modification of a fourth-order tensor is defined by a formula similar in structure to Eq. (14.72). The fourth-order inverse is given by a formula similar to Eq. (14.73). Namely,

$$\text{If} \quad \underset{\approx}{B} = \underset{\approx}{A} + \underset{\sim}{V}\underset{\sim}{W} \qquad (B_{ijkl} = A_{ijkl} + V_{ij}W_{kl}), \tag{17.19}$$

then

$$\underset{\approx}{\mathbf{B}}^{-1} = \underset{\approx}{\mathbf{A}}^{-1} - \frac{\underset{\approx}{\mathbf{A}}^{-1} : \underset{\sim}{\mathbf{V}}\underset{\sim}{\mathbf{W}} : \underset{\approx}{\mathbf{A}}^{-1}}{1 + \underset{\sim}{\mathbf{W}} : \underset{\approx}{\mathbf{A}}^{-1} : \underset{\sim}{\mathbf{V}}} \qquad \left(B^{-1}_{ijkl} = A^{-1}_{ijkl} - \frac{A^{-1}_{ijmn} V_{mn} W_{rs} A^{-1}_{rskl}}{1 + W_{pq} A^{-1}_{pqtu} V_{tu}} \right). \tag{17.20}$$

Structurally, this fourth-order formula is identical to the second-order formula except that the vector inner products (single dot) are replaced with tensor inner products (double dot). The Sherman-Morrison formula in Eq. (17.20) is frequently used in plasticity theory.

Exercise 17.1 Directly substitute Eq. (17.20) into $\underset{\approx}{\mathbf{B}}^{-1} : \underset{\approx}{\mathbf{B}}$ to confirm that the result is indeed the identity and that it applies even if $\underset{\approx}{\mathbf{A}}$ is not major symmetric.

Hint: The answer needs to reduce to $\underset{\approx}{\mathbf{A}}^{-1} : \underset{\approx}{\mathbf{A}}$, which is the fourth-order identity by premise.

Inverse of a special symmetric rank-2 modification

In materials modeling of deformation-induced elastic anisotropy, one encounters the need to invert a tensor of the form

$$\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{C}} + \underset{\sim}{\mathbf{V}}\underset{\sim}{\mathbf{W}} + \underset{\sim}{\mathbf{W}}\underset{\sim}{\mathbf{V}} \tag{17.21}$$

where $\underset{\approx}{\mathbf{C}}$ is major symmetric and the second-order tensors $\underset{\sim}{\mathbf{V}}$ and $\underset{\sim}{\mathbf{W}}$ are multiplied dyadically in the last two terms (*i.e.*, $B_{ijkl} = C_{ijkl} + V_{ij} W_{kl} + W_{ij} V_{kl}$. Referring to the analogous task in Eqs. (14.84) through (14.88), the answer for the inverse is

$$\underset{\approx}{\mathbf{B}}^{-1} = \underset{\approx}{\mathbf{C}}^{-1} - \frac{\underset{\approx}{\mathbf{C}}^{-1} : [(1 + X_{vw})(\underset{\sim}{\mathbf{V}}\underset{\sim}{\mathbf{W}} + \underset{\sim}{\mathbf{W}}\underset{\sim}{\mathbf{V}}) - X_{vv}\underset{\sim}{\mathbf{W}}\underset{\sim}{\mathbf{W}} - X_{ww}\underset{\sim}{\mathbf{V}}\underset{\sim}{\mathbf{V}}] : \underset{\approx}{\mathbf{C}}^{-1}}{(1 + X_{vw})^2 - X_{ww} X_{vv}}, \tag{17.22}$$

where

$$X_{vv} = \underset{\sim}{\mathbf{V}} : \underset{\approx}{\mathbf{C}}^{-1} : \underset{\sim}{\mathbf{V}} \qquad X_{vw} = \underset{\sim}{\mathbf{V}} : \underset{\approx}{\mathbf{C}}^{-1} : \underset{\sim}{\mathbf{W}}$$

$$X_{wv} = \underset{\sim}{\mathbf{W}} : \underset{\approx}{\mathbf{C}}^{-1} : \underset{\sim}{\mathbf{V}} \qquad X_{ww} = \underset{\sim}{\mathbf{W}} : \underset{\approx}{\mathbf{C}}^{-1} : \underset{\sim}{\mathbf{W}} \tag{17.23}$$

Special case:

Suppose that $\underset{\sim}{\mathbf{V}}$ and $\underset{\sim}{\mathbf{W}}$ happen to be eigentensors of $\underset{\approx}{\mathbf{C}}$ corresponding to eigenvalues λ_v and λ_w. Then

$$X_{vv} = \frac{v^2}{\lambda_v} \qquad X_{vw} = 0, \qquad \text{where } v^2 = \underset{\sim}{\mathbf{V}} : \underset{\sim}{\mathbf{V}}$$

$$X_{wv} = 0 \qquad X_{ww} = \frac{w^2}{\lambda_w}, \qquad \text{where } w^2 = \underset{\sim}{\mathbf{W}} : \underset{\sim}{\mathbf{W}} \tag{17.24}$$

and the inverse is therefore given by

$$\underset{\approx}{\underset{\approx}{B}}^{-1} = \underset{\approx}{\underset{\approx}{C}}^{-1} - \frac{\left[(\underset{\approx}{V}\underset{\approx}{W} + \underset{\approx}{W}\underset{\approx}{V}) - \frac{v^2}{\lambda_w}\underset{\approx}{W}\underset{\approx}{W} - \frac{w^2}{\lambda_v}\underset{\approx}{V}\underset{\approx}{V}\right]}{\lambda_v\lambda_w - w^2v^2}. \tag{17.25}$$

Having a fourth-order tensor is often less useful than knowing the result when it acts on an arbitrary second-order tensor $\underset{\approx}{X}$. For this special case where $\underset{\approx}{V}$ and $\underset{\approx}{W}$ are eigentensors of $\underset{\approx}{\underset{\approx}{C}}$, the action of the tensor $\underset{\approx}{\underset{\approx}{B}}$ and its inverse on an arbitrary second-order tensor $\underset{\approx}{X}$ are

$$\underset{\approx}{\underset{\approx}{B}}:\underset{\approx}{X} = \underset{\approx}{\underset{\approx}{C}}:\underset{\approx}{X} + (x_w\underset{\approx}{V} + x_v\underset{\approx}{W}) \tag{17.26a}$$

$$\underset{\approx}{\underset{\approx}{B}}^{-1}:\underset{\approx}{X} = \underset{\approx}{\underset{\approx}{C}}^{-1}:\underset{\approx}{X} - (\alpha_v\underset{\approx}{V} + \alpha_w\underset{\approx}{W}) \tag{17.26b}$$

where

$$x_v = \underset{\approx}{X}:\underset{\approx}{V} \qquad \text{and} \qquad x_w = \underset{\approx}{X}:\underset{\approx}{W}, \tag{17.27}$$

and the α-coefficients are

$$\alpha_v = \frac{\left[x_w - \frac{w^2}{\lambda_v}x_v\right]}{\lambda_v\lambda_w - w^2v^2} \qquad \text{and} \qquad \alpha_w = \frac{\left[x_v - \frac{v^2}{\lambda_w}x_w\right]}{\lambda_v\lambda_w - w^2v^2}. \tag{17.28}$$

In materials modeling of recoverable deformation-induced anisotropy of an isotropic material (*cf. Fuller-Brannon citation*), $\underset{\approx}{\underset{\approx}{C}}$ is the small-strain *isotropic* stiffness, $\underset{\approx}{V}$ is a multiple of the identity tensor, and $\underset{\approx}{W}$ is the strain deviator (hence making them eigentensors of $\underset{\approx}{\underset{\approx}{C}}$). The rank-2 modification in Eq. (17.21) approximates the effect of induced anisotropy by making the material become stiffer in the stretching direction defined by $\underset{\approx}{W}$. Note from Eq. (17.21) that the anisotropic stiffness $\underset{\approx}{\underset{\approx}{B}}$ is identical to the nominal isotropic stiffness $\underset{\approx}{\underset{\approx}{C}}$ whenever the strain deviator $\underset{\approx}{W}$ is zero. Also note that, since $\underset{\approx}{V}$ is isotropic and $\underset{\approx}{W}$ is deviatoric, these tensors are not only orthogonal to each other (*i.e.,* $\underset{\approx}{V}:\underset{\approx}{W} = 0$), but they are also eigentensors of $\underset{\approx}{\underset{\approx}{C}}$. The corresponding eigenvalues are denoted $3K$ and $2G$, where K is the bulk modulus and G is the shear modulus. Thus, thermodynamically necessary induced anisotropy can be added to an existing thermodynamically *inadmissible* code by simply adding two simple terms in parentheses in Eqs. (17.26). See Exercise 27.6 for details.

Higher-order tensor inner product

As an engineering convenience (akin to denoting time rates with superposed dots), we have defined all of our inner products such that the number of "dots" indicates the number of contracted indices. Clearly, this notation is not practical for higher-order tensors. An alternative notation for an n^{th}-order inner product may be defined as the order n surrounded by a circle. Thus, for example,

$$\underset{\underset{\approx}{\approx}}{\boldsymbol{X}} \; ④ \; \underset{\underset{\approx}{\approx}}{\boldsymbol{Y}} \text{ means the same thing as } \underset{\underset{\approx}{\approx}}{\boldsymbol{X}} :: \underset{\underset{\approx}{\approx}}{\boldsymbol{Y}}. \tag{17.29}$$

Some writers [e.g., Ref. 60*] prefer always using a single raised dot to denote all inner-products, regardless of the order. These writers demand that meaning of the single-dot operator must be inferred by the tensorial order of the arguments. The reader is further expected to infer the tensorial order of the arguments from the context of the discussion since most writers do not indicate tensor order by the number of under-tildes. These writers tend to define the multiplication of two tensors written side by side (with no multiplication symbol between them) to be the tensor composition. For example, when they write $\boldsymbol{AB}$ between two tensors that have been identified as second-order, then they mean what we would write as $\underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{B}}$. When they write $\boldsymbol{UV}$ between two tensors that have been identified as fourth-order, they mean what we would write as $\underset{\underset{\approx}{\approx}}{\boldsymbol{U}} : \underset{\underset{\approx}{\approx}}{\boldsymbol{V}}$. Such notational conventions are undeniably easier to typeset, and they work fine whenever one restricts attention to the small set of conventional tensor operations normally seen in trivial applications. However, more exotic advanced tensor operations become difficult to define under this system. A consistent self-defining system such as the one used in this book is far more convenient and flexible.

Self-defining notation

Throughout this book, our notation is self-defining in the sense that the meaning of an expression can always be ascertained by expanding all arguments in basis form, as discussed on page 384. The following list shows several indicial expressions along with their direct notation expressions under our notation

$U_{mnpq}V_{mnpq}$	$\underset{\underset{\approx}{\approx}}{\boldsymbol{U}} :: \underset{\underset{\approx}{\approx}}{\boldsymbol{V}}$
$U_{ijpq}V_{pqkl}$	$\underset{\underset{\approx}{\approx}}{\boldsymbol{U}} : \underset{\underset{\approx}{\approx}}{\boldsymbol{V}}$

* We call attention to this reference not because it is the only example, but because it a continuum mechanics textbook that is in common use today and may therefore be familiar to a larger audience. This notation draws from older classic references [e.g., 47]. Older should not always be taken to mean inferior, but we believe that, in this case, the older tensor notation is needlessly flawed for engineering purposes. Our notation requires a different symbol for inner products on differently ordered tensor spaces, whereas the older style overloads *the same symbol* to mean different inner products — operator overloading can be extremely useful in many situations, but we feel it does more harm than good in this case because it precludes self-defining notation.

$$U_{ijkp}V_{plmn} \qquad \underset{\approx}{\underset{\approx}{U}} \bullet \underset{\approx}{\underset{\approx}{V}}$$

$$U_{ijkl}V_{mnpq} \qquad \underset{\approx}{\underset{\approx}{U}}\underset{\approx}{\underset{\approx}{V}}$$

$$A_{pq}U_{pqij} \qquad \underset{\approx}{A}:\underset{\approx}{\underset{\approx}{U}}$$

$$A_{ip}U_{pjkl} \qquad \underset{\approx}{A} \bullet \underset{\approx}{\underset{\approx}{U}}$$

$$A_{ij}U_{klmn} \qquad \underset{\approx}{A}\underset{\approx}{\underset{\approx}{U}}$$

$$A_{ip}\varepsilon_{pjq}U_{qklm} \qquad -\underset{\approx}{A} \times \underset{\approx}{\underset{\approx}{U}}^{*} \tag{17.30}$$

Writers who use inconsistent non-self-defining notational structures would be hard-pressed to come up with easily remembered direct notations for all of the above operations. Their only recourse would be to meekly argue that such operations would never be needed in real applications anyway. Before we come off sounding too pompous, we acknowledge that there exist indicial expressions that do not translate elegantly into our system. For example, the equation

$$A_{ijkl}\underset{\sim}{e}_i\underset{\sim}{e}_j\underset{\sim}{e}_k\underset{\sim}{e}_l = U_{ikjl}\underset{\sim}{e}_i\underset{\sim}{e}_j\underset{\sim}{e}_k\underset{\sim}{e}_l \tag{17.31}$$

would have to be written under our notational system as

$$\underset{\approx}{\underset{\approx}{A}} = X_2^3\left(\underset{\approx}{\underset{\approx}{U}}\right), \tag{17.32}$$

where the rather non-intuitive swap operator X_2^3 is defined in Eq. (25.51). Of course, older notation systems have no commonly recognized direct notation for this operation either. This particular operation occurs so frequently that we later (page 472) introduce a new "leafing" operator to denote it by $\underset{\approx}{\underset{\approx}{A}} = \underset{\approx}{\underset{\approx}{U}}^L$ as an alternative to Eq. (17.32). Even when using the notational scheme that we advocate, writers should always provide indicial expressions to clarify their notations, especially when the operations are rather unusual.

The difficulties with direct notation might seem to suggest that perhaps indicial notation would be the best choice. In some instances, this is true. However, even indicial notation has its pitfalls, principally in operator precedence. For example, the notation

$$\frac{\partial f}{\partial \sigma_{kk}} \tag{17.33}$$

is ambiguous. It could be interpreted in two different ways:

* In this equation, the negative appears because the cross product is defined such that the summed indices on the alternating symbol must be adjacent (making them adjacent involves a negative permutation of ε_{pjq} to make it $-\varepsilon_{jpq}$.

$$\frac{\partial f}{\partial(\mathrm{tr}\,\underset{\approx}{\sigma})} \quad \text{or} \quad \mathrm{tr}\left(\frac{\partial f}{\partial \underset{\approx}{\sigma}}\right) \tag{17.34}$$

These two operations give different results. Furthermore, we have already seen that the book-keeping needed to satisfy the summation conventions is tedious, error-prone, often limited to Cartesian components, distracting from general physical interpretations, and (in some cases) not well-suited to calculus manipulations. Nonetheless, there are certainly many instances where indicial notation is the most lucid choice, so be flexible.

Bottom line: in your own work, use the notation you prefer, but in published and presented work, always employ notation that is likely to achieve the interpretation of your work that you desire from educated readers. Your goal is to convince them of the truth of a scientific principle, not to intimidate, condescend, or baffle them with your (or our) whacked out notation.

The magnitude of a tensor or a vector

The magnitude of a second-order tensor $\underset{\approx}{A}$ is a scalar denoted $\|\underset{\approx}{A}\|$ defined

$$\|\underset{\approx}{A}\| \equiv \sqrt{\underset{\approx}{A}:\underset{\approx}{A}}\,. \tag{17.35}$$

This definition has exactly the same form as the more familiar definition of the **magnitude of a simple vector** $\boldsymbol{v}$:

$$\|\underset{\sim}{v}\| \equiv \sqrt{\underset{\sim}{v} \bullet \underset{\sim}{v}}\,. \tag{17.36}$$

Though rarely needed, the magnitude of a fourth-order tensor $\underset{\approx\approx}{X}$ is a scalar defined

$$\|\underset{\approx\approx}{X}\| = \sqrt{\underset{\approx\approx}{X}::\underset{\approx\approx}{X}}\,. \tag{17.37}$$

A vector is zero if and only if its magnitude is zero. Likewise, a tensor is zero if and only if its magnitude is zero.

Useful inner product identities

The symmetry and deviator decompositions of tensors are often used in conjunction with the following identities:

$$\underset{\approx}{A}:\underset{\approx}{B} = \mathrm{sym}\underset{\approx}{A}:\mathrm{sym}\underset{\approx}{B} + \mathrm{skw}\underset{\approx}{A}:\mathrm{skw}\underset{\approx}{B} \tag{17.38}$$

$$\underset{\approx}{A}:\underset{\approx}{B} = \mathrm{dev}\underset{\approx}{A}:\mathrm{dev}\underset{\approx}{B} + \mathrm{iso}\underset{\approx}{A}:\mathrm{iso}\underset{\approx}{B}\,. \tag{17.39}$$

Decomposing a tensor into its symmetric plus skew symmetric parts ($\underset{\approx}{A} = \mathrm{sym}\underset{\approx}{A} + \mathrm{skw}\underset{\approx}{A}$ and $\underset{\approx}{B} = \mathrm{sym}\underset{\approx}{B} + \mathrm{skw}\underset{\approx}{B}$) represents an orthogonal projection decomposition that is completely analogous to Eq. (15.21). Thus, Eq. (17.38) is a specific application of Eq. (15.23) in which tensors are interpreted in their V_9^1 sense. A similar statement holds for the decomposition of tensors into deviatoric plus isotropic parts. When applied to the special case $\underset{\approx}{A}:\underset{\approx}{A}$, the above formulas are tensor versions of the Pythagorean formula.

If $\underset{\approx}{\boldsymbol{B}}$ happens to be a symmetric tensor (*i.e.,* if $\operatorname{skw}\underset{\approx}{\boldsymbol{B}} = \underset{\approx}{\boldsymbol{0}}$) then the inner product between $\underset{\approx}{\boldsymbol{B}}$ any other tensor $\underset{\approx}{\boldsymbol{A}}$ will depend only on the symmetric part of $\underset{\approx}{\boldsymbol{A}}$. Consequently, sometimes researchers will replace $\underset{\approx}{\boldsymbol{A}}$ by its symmetric part *without any loss in generality* — which can save on storage in numerical computations, but is unwise if there is any chance that $\underset{\approx}{\boldsymbol{A}}$ will need to be used in any other context.

Incidentally, note that the "trace" operation defined in Eq. (13.71) can be written as an inner product inner product with the identity tensor:

$$\boxed{\operatorname{tr}\underset{\approx}{\boldsymbol{A}} \equiv \underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{A}}}\,. \tag{17.40}$$

Also note that $\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{I}} = \operatorname{tr}\underset{\approx}{\boldsymbol{I}} = 3$, so Eq. (17.39) may be alternatively written

$$\boxed{\underset{\approx}{\boldsymbol{A}}:\underset{\approx}{\boldsymbol{B}} = \underset{\approx}{\boldsymbol{A}}':\underset{\approx}{\boldsymbol{B}}' + \frac{1}{3}(\operatorname{tr}\underset{\approx}{\boldsymbol{A}})(\operatorname{tr}\underset{\approx}{\boldsymbol{B}})}\,. \tag{17.41}$$

Distinction between an Nth-order tensor and an Nth-rank tensor

Many authors use the term "N^{th}-rank tensor" to mean what we would call an "N^{th}-order tensor." We don't adopt this practice because the term "rank" has a specific meaning in matrix analysis that applies equally well for tensor analysis. We would *prefer* the "rank" of a second-order tensor to be defined as equaling the rank of its Cartesian component matrix (i.e., the number of linearly independent rows or columns). Of course, our practice of saying N^{th}-*order* tensors has its downside too because it can cause confusion when discussing tensor polynomials. Since many authors use the word "rank" to mean what we refer to as "order," the best strategy is to not use the word "rank" at all. When we later speak of type-1 projectors, for example, they will be seen to have a matrix rank of 1.

When a second-order tensor is regarded as an operation that takes vectors to vectors, then the "matrix rank" of the second-order tensor is the dimension of the range space. For example, if a second-order tensor projects a vector into its part in the direction of some fixed unit vector, then the operation always outputs a vector that is a multiple of the fixed unit vector, making the operator's range space one-dimensional and its matrix rank 1. A type-2 second-order projector is a tensor that projects vectors to a 2-dimensional space, and these will later be seen to have a matrix rank of 2. The identity tensor has a matrix rank of 3. Based on well-known matrix theory, a second-order engineering tensor is invertible only if its matrix rank is 3.

Fourth-order oblique tensor projections

Second-order tensors are themselves 9-dimensional abstract vectors of type V_9^1 with "$:$" denoting the inner product. Consequently, operations that are defined for ordinary 3D vectors have analogs for tensors. Recall that Eq. (11.29) gave the formula for the oblique projection of a vector $\underset{\sim}{\boldsymbol{x}}$ onto a plane perpendicular to a given vector $\underset{\sim}{\boldsymbol{b}}$. The "light rays" defining the projection direction were parallel to the vector $\underset{\sim}{\boldsymbol{a}}$. The analog of Eq. (11.29) for tensors is

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

TENSOR ALGEBRA

$$P(\underset{\approx}{\mathbf{X}}) = \underset{\approx}{\mathbf{X}} - \frac{\underset{\approx}{\mathbf{A}}(\underset{\approx}{\mathbf{B}}:\underset{\approx}{\mathbf{X}})}{\underset{\approx}{\mathbf{A}}:\underset{\approx}{\mathbf{B}}}. \tag{17.42}$$

As was the case for the projection in 3-space, this operation represents a linear oblique projection in tensor space. The "hypersurface" to which $\underset{\approx}{\mathbf{X}}$ is projected is perpendicular to $\underset{\approx}{\mathbf{B}}$ and the oblique projection direction is aligned with $\underset{\approx}{\mathbf{A}}$. This projection function appears in the study of plasticity (*cf.* [70, 83,16]) in which a trial stress state is returned to the yield surface via a projection of this form. The return path is oblique with respect to an ordinary notion of distance, or "closest point" with respect to a different (energy-norm) definition of distance [83].

The fourth-order projection transformation can be readily verified to have the following properties:

$$P(\alpha\underset{\approx}{\mathbf{X}}) = \alpha P(\underset{\approx}{\mathbf{X}}) \text{ for all scalars } \alpha. \tag{17.43}$$

$$P(\underset{\approx}{\mathbf{X}}+\underset{\approx}{\mathbf{Y}}) = P(\underset{\approx}{\mathbf{X}})+P(\underset{\approx}{\mathbf{Y}}) \text{ for all } \underset{\approx}{\mathbf{X}} \text{ and } \underset{\approx}{\mathbf{Y}}. \tag{17.44}$$

$$P(P(\underset{\approx}{\mathbf{X}})) = P(\underset{\approx}{\mathbf{X}}). \tag{17.45}$$

The first two properties simply indicate that the projection operation is linear. The last property says that projecting a tensor that has already been projected merely gives the tensor back unchanged.

Finally, the analog of Eqs. (11.47) and (11.48) is the important identity that

$$P(\underset{\approx}{\mathbf{X}}) = P(\underset{\approx}{\mathbf{Y}}) \quad \text{if and only if} \quad \underset{\approx}{\mathbf{X}} = \underset{\approx}{\mathbf{Y}}+\beta\underset{\approx}{\mathbf{A}}. \tag{17.46}$$

This identity is used, for example, to prove the validity of radial return algorithms in plasticity theory [16].

Leafing and palming operations

Consider a deck of cards. If there are an even number of cards, you can split the deck in half and (in principle) leaf the cards back together in a perfect shuffle. We would call this a leafing operation. If, for example, there were six cards in the deck initially ordered sequentially, then, after the leafing operation (perfect shuffle), they would be ordered 142536. If the deck had only four cards, they would leaf into the ordering 1324.

We will here define a similar operation that applies to any even order tensor. The structure to indicate application of this leafing operation will be a superscript "L." Let

$$\underset{\approx}{\mathbf{U}} = U_{ijpq}\underset{\sim}{\mathbf{e}}_i\underset{\sim}{\mathbf{e}}_j\underset{\sim}{\mathbf{e}}_p\underset{\sim}{\mathbf{e}}_q \tag{17.47}$$

denote a fourth-order tensor. The "leaf" of this tensor will be defined

$$\underset{\approx}{\mathbf{U}}^L = U_{ipjq}\underset{\sim}{\mathbf{e}}_i\underset{\sim}{\mathbf{e}}_j\underset{\sim}{\mathbf{e}}_p\underset{\sim}{\mathbf{e}}_q. \tag{17.48}$$

TENSOR ALGEBRA

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

As seen here, the leaf entails a permutation of the indices on U_{ijpq} as if `ijpq` were a deck of four cards split into two parts (`ij` and `pq`) that are recombined in a perfect shuffle. In purely indicial notation, we would write

$$U^L_{ijpq} = U_{ipjq}. \tag{17.49}$$

To remember this equation, you *could* just say that the middle two indices are swapped, but it is better to think of the indices as being distributed in a manner that alternates back and fourth between the first and last halves of the index groups: `i` is in the first half, then `j` is in the second half, followed by `p` back in the first half and `q` in the second half. This way of thinking about a leafing operation makes it easier to extend to higher-order tensors, as discussed below.

Note that shuffling the indices in Eq. (17.48) is equivalent to shuffling the dyadic ordering of the basis vectors. In other words, the equation

$$\underset{\approx}{U} = U_{ijpq}\underset{\sim}{e}_i\underset{\sim}{e}_p\underset{\sim}{e}_j\underset{\sim}{e}_q \tag{17.50}$$

is equivalent to Eq. (17.48). This interpretation of the operation allows it to be easily generalized to non-orthonormal bases.

Derivative of a leafing operation:

$$\frac{\partial U^L_{ijpq}}{\partial U_{mnrs}} = \frac{\partial U_{ipjq}}{\partial U_{mnrs}} = \delta_{im}\delta_{pn}\delta_{jr}\delta_{qs} = \delta_{im}\delta_{jr}\delta_{pn}\delta_{qs}. \tag{17.51}$$

The leaf of a sixth-order tensor with components U_{ijkpqr} would be

$$U^L_{ijkpqr} = U_{ipjqkr}. \tag{17.52}$$

The leaf of a second-order tensor with components U_{ij} would be

$$U^L_{ij} = U_{ij}, \tag{17.53}$$

which causes no rearrangement of indices and is therefore of no interest.

Now consider a different way to shuffle a deck of cards, which might be used by a "cheater." First the deck is split in half, but then the second half is reversed before shuffling. For example, a six-card deck, originally ordered 123456 would split into halves 123 and 456. After reversing the order of the second half, the halves would be 123 654, and then shuffling would give 162534. We will call the analog of this operation on tensor indices a "palming" operation and denote it with a superscript Γ (i.e., an upside down "L"). Then, for fourth- and sixth-order tensors, the palming operator would give

$$U^\Gamma_{ijkl} = U_{iljk}, \tag{17.54}$$

and

$$U^\Gamma_{ijkpqr} = U_{irjqkp}. \tag{17.55}$$

The leafing and palming operations have been introduced simply because these types of index re-orderings occur frequently in higher-order analyses, and there is no straightforward way to characterize them in conventional direct structural notation. Using these new operations, note that the e-δ identity can be written

$$\underset{\approx}{\varepsilon} \bullet \underset{\approx}{\varepsilon} = (\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})^{\mathrm{L}} - (\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})^{\Gamma} . \tag{17.56}$$

Here, $\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}}$ is a dyad so that $(\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})_{ijmn} = \delta_{ij}\delta_{mn}$ and therefore $(\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})^{L}_{ijmn} = \delta_{im}\delta_{jn}$ and $(\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})^{\Gamma}_{ijmn} = \delta_{in}\delta_{jm}$.

Some writers [*cf.,* 32] define a "cross-composition" operator "^" by

$$\underset{\approx}{\boldsymbol{A}} \wedge \underset{\approx}{\boldsymbol{B}} \equiv \frac{1}{2}(A_{im}B_{jn} + A_{in}B_{jm})\underset{\sim}{\boldsymbol{e}}_i\underset{\sim}{\boldsymbol{e}}_j\underset{\sim}{\boldsymbol{e}}_m\underset{\sim}{\boldsymbol{e}}_n , \tag{17.57}$$

which can be written in terms of leaf and palm operators as

$$\underset{\approx}{\boldsymbol{A}} \wedge \underset{\approx}{\boldsymbol{B}} = \frac{1}{2}[(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^{\mathrm{L}} + (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^{\Gamma}] \tag{17.58}$$

An important special case is

$$\underset{\approx}{\boldsymbol{I}} \wedge \underset{\approx}{\boldsymbol{I}} = \underset{\approx}{\boldsymbol{P}}^{\mathrm{sym}} = \text{symmetry projection operator} \tag{17.59}$$

Important application of the leaf operator. The operation $\underset{\approx}{\boldsymbol{Y}} = \underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{B}}^T$ is linear with respect to $\underset{\approx}{\boldsymbol{X}}$, so the representation theorem guarantees existence of a fourth-order tensor $\underset{\approx}{\boldsymbol{U}}$ such that $\underset{\approx}{\boldsymbol{Y}} = \underset{\approx}{\boldsymbol{U}}{:}\underset{\approx}{\boldsymbol{X}}$. This fourth-order tensor is, in fact, $(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^L$. Thus

$$\boxed{\underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{B}}^T = (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^{\mathrm{L}}{:}\underset{\approx}{\boldsymbol{X}}} . \tag{17.60}$$

Here we have considered the operation $\underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{B}}^T$ because, in applications, it may be interpreted as a transformation of each basis vector in $\underset{\approx}{\boldsymbol{X}}$. Specifically,

$$\underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{B}}^T = X_{ij}\ (\underset{\approx}{\boldsymbol{A}} \bullet \underset{\sim}{\boldsymbol{e}}_i)\ (\underset{\approx}{\boldsymbol{B}} \bullet \underset{\sim}{\boldsymbol{e}}_j) . \tag{17.61}$$

Note: in this form, $\underset{\approx}{\boldsymbol{B}}$ is *not* transposed

In other words, $\underset{\approx}{\boldsymbol{A}}$ transforms the first basis vector in $X_{ij}\underset{\sim}{\boldsymbol{e}}_i\underset{\sim}{\boldsymbol{e}}_j$, while $\underset{\approx}{\boldsymbol{B}}$ (not its transpose) acts on the second basis vector. In Eq. (17.60), there is a transpose on $\underset{\approx}{\boldsymbol{B}}$ to move it from being "stuck in the middle" of Eq. (17.61) by using Eq. (13.45). Incidentally, the importance of $(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^L$ was recognized by Del Piero [28], who typeset the operation as $\underset{\approx}{\boldsymbol{A}} \boxtimes \underset{\approx}{\boldsymbol{B}}$. This "**square tensor product**" notation, sometimes referred to as the **Kronecker product**, has also been adopted in more recent work [*cf.* 58], but we will simply write $(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^L$. The merits of $(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^L$ over $\underset{\approx}{\boldsymbol{A}} \boxtimes \underset{\approx}{\boldsymbol{B}}$ become clearer in an upcoming section on symmetrized leafing, which applies when (as is common in continuum mechanics) $\underset{\approx}{\boldsymbol{X}}$ is restricted to being symmetric. This and other common notations for special product operators are summarized below:

$\underset{\approx}{\mathbf{A}} \otimes \underset{\approx}{\mathbf{B}}$ (by other authors) means the same thing as our $\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}}$. Therefore,

$$[\underset{\approx}{\mathbf{A}} \otimes \underset{\approx}{\mathbf{B}}]\boldsymbol{:}\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{A}}(\underset{\approx}{\mathbf{B}}\boldsymbol{:}\underset{\approx}{\mathbf{C}}), \text{ and thus } [\underset{\approx}{\mathbf{A}} \otimes \underset{\approx}{\mathbf{B}}]_{ijkl} = A_{ij}B_{kl}. \tag{17.62}$$

$\underset{\approx}{\mathbf{A}} \boxtimes \underset{\approx}{\mathbf{B}}$ (by other authors) means the same thing as our $(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^L$, Therefore

$$[\underset{\approx}{\mathbf{A}} \boxtimes \underset{\approx}{\mathbf{B}}]\boldsymbol{:}\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{A}} \bullet \underset{\approx}{\mathbf{C}} \bullet \underset{\approx}{\mathbf{B}}^T, \text{ and thus } [\underset{\approx}{\mathbf{A}} \boxtimes \underset{\approx}{\mathbf{B}}]_{ijkl} = A_{ik}B_{jl}. \tag{17.63}$$

$\underset{\approx}{\mathbf{A}} \underline{\otimes} \underset{\approx}{\mathbf{B}}$ (by other authors) means the same thing as our $(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^L$ applied after a symmetry projection. Therefore, $\underset{\approx}{\mathbf{A}} \underline{\otimes} \underset{\approx}{\mathbf{B}} = \frac{1}{2}[(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^L + (\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^\Gamma]$. That is,

$$[\underset{\approx}{\mathbf{A}} \underline{\otimes} \underset{\approx}{\mathbf{B}}]\boldsymbol{:}\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{A}} \bullet \operatorname{sym}\underset{\approx}{\mathbf{C}} \bullet \underset{\approx}{\mathbf{B}}^T, \text{ and thus } [\underset{\approx}{\mathbf{A}} \underline{\otimes} \underset{\approx}{\mathbf{B}}]_{ijkl} = \tfrac{1}{2}(A_{ik}B_{jl} + A_{il}B_{jk}). \tag{17.64}$$

Some pertinent properties are

$$[\underset{\approx}{\mathbf{A}} \boxtimes \underset{\approx}{\mathbf{B}}]:[\underset{\approx}{\mathbf{C}} \boxtimes \underset{\approx}{\mathbf{D}}] = (\underset{\approx}{\mathbf{A}} \bullet \underset{\approx}{\mathbf{C}}) \boxtimes (\underset{\approx}{\mathbf{B}} \bullet \underset{\approx}{\mathbf{D}}) \tag{17.65}$$

$$[\underset{\approx}{\mathbf{A}} \boxtimes \underset{\approx}{\mathbf{B}}]^{-1} = \underset{\approx}{\mathbf{A}}^{-1} \boxtimes \underset{\approx}{\mathbf{B}}^{-1} \tag{17.66}$$

$$[\underset{\approx}{\mathbf{A}} \underline{\otimes} \underset{\approx}{\mathbf{B}}]:[\underset{\approx}{\mathbf{C}} \underline{\otimes} \underset{\approx}{\mathbf{D}}] = \frac{1}{2}[(\underset{\approx}{\mathbf{A}} \bullet \underset{\approx}{\mathbf{C}}) \underline{\otimes} (\underset{\approx}{\mathbf{B}} \bullet \underset{\approx}{\mathbf{D}}) + (\underset{\approx}{\mathbf{A}} \bullet \underset{\approx}{\mathbf{D}}) \underline{\otimes} (\underset{\approx}{\mathbf{B}} \bullet \underset{\approx}{\mathbf{C}})] \tag{17.67}$$

Major transpose of a fourth-order tensor

Given a fourth-order tensor U_{ijkl}, the major transpose is defined

$$U^T_{ijkl} = U_{klij} \tag{17.68}$$

If the tensor is major-symmetric, it equals its own major transpose (and vice versa). Some commonly encountered properties are

$$(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^T = \underset{\approx}{\mathbf{B}}\underset{\approx}{\mathbf{A}} \tag{17.69a}$$

$$(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^{LT} = (\underset{\approx}{\mathbf{A}}^T\underset{\approx}{\mathbf{B}}^T)^L \tag{17.69b}$$

$$(\underset{\approx}{\mathbf{A}}\underset{\approx}{\mathbf{B}})^{\Gamma T} = (\underset{\approx}{\mathbf{B}}^T\underset{\approx}{\mathbf{A}}^T)^\Gamma \tag{17.69b}$$

(17.69c)

Minor-symmetrizing a fourth-order tensor

Given a fourth-order tensor U_{ijkl}, a common operation in materials modeling involves *minor-symmetrizing* the minor indices. Just as a tensor A_{ij} can be symmetrized by $\frac{1}{2}(A_{ij} + A_{ji})$, a fourth-order tensor can be minor-symmetrized by

$$U^{\sigma}_{ijkl} = U_{(ij)(kl)} = \tfrac{1}{4}(U_{ijkl} + U_{jikl} + U_{ijlk} + U_{jilk}). \tag{17.70}$$

Here, we have employed a common indicial notation convention that pairs of indices in parentheses are to be expanded in a symmetric sum. With this convention, for example, $A^{\text{sym}}_{ij} = A_{(ij)}$. In symbolic notation,

$$\underset{\approx}{U}^{\sigma} = \underset{\approx}{P}^{\text{sym}} : \underset{\approx}{U} : \underset{\approx}{P}^{\text{sym}} \tag{17.71}$$

The minor-symmetrization does not commute with the leaf operator in Eq. (17.48). In other words,

$$\underset{\approx}{U}^{L\sigma} \neq \underset{\approx}{U}^{\sigma L} \text{ in general.} \tag{17.72}$$

Major-symmetrizing a fourth-order tensor

The operation of major-symmetrizing a fourth-order tensor will be denoted with a superscript "M" so that

$$U^{M}_{ijkl} = \tfrac{1}{2}(U_{ijkl} + U_{klij}) \tag{17.73}$$

The major symmetrization operation is linear, which means it commutes with linear combinations. Some other properties are

$$\underset{\approx}{U}^{LM} \neq \underset{\approx}{U}^{ML} \text{ in general, but...} \tag{17.74}$$

$$\underset{\approx}{U}^{LM} = \underset{\approx}{U}^{ML} \text{ if } \underset{\approx}{U} \text{ is both major and minor symmetric.} \tag{17.75}$$

$$\underset{\approx}{U}^{\Gamma M} \neq \underset{\approx}{U}^{M\Gamma} \text{ in general, but...} \tag{17.76}$$

$$\underset{\approx}{U}^{\Gamma M} = \underset{\approx}{U}^{M\Gamma} \text{ if } \underset{\approx}{U} \text{ is both major and minor symmetric.} \tag{17.77}$$

$$\underset{\approx}{U}^{\sigma M} = \underset{\approx}{U}^{M\sigma} \tag{17.78}$$

Symmetrized Leafing

Consider a leafed tensor $U^{L}_{ijkl} = U_{ikjl}$. Even if the tensor U_{ijkl} is minor-symmetric, its leaf will not necessarily be minor symmetric. The symmetrized leaf is denoted with a superscript λ and defined by first leafing via the leaf "L" operator in Eq. (17.48) and then symmetrizing via the "σ" operator in Eq. (17.70):

$$U^{\lambda}_{ijkl} = U^{L\sigma}_{ijkl} = \frac{1}{4}(U_{ikjl} + U_{iklj} + U_{kijl} + U_{kilj}). \tag{17.79}$$

In symbolic form, the symmetrized leaf may be defined as

$$\underset{\approx}{U}^{\lambda} = \underset{\approx}{P}^{\text{sym}} : \underset{\approx}{U}^{L} : \underset{\approx}{P}^{\text{sym}} = \underset{\approx}{U}^{L\sigma}, \tag{17.80}$$

where $\underset{\approx}{P}^{\text{sym}}$ is the symmetry operator defined in Eq. 14.14.

Specifically, the Kronecker product (i.e., the leaf of a tensor-tensor dyad) tends to show up in applications in the a symmetric form such as

$$\underset{\approx}{\boldsymbol{Y}} = \underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{B}}^T + \underset{\approx}{\boldsymbol{B}} \bullet \underset{\approx}{\boldsymbol{X}} \bullet \underset{\approx}{\boldsymbol{A}}^T = (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}} + \underset{\approx}{\boldsymbol{B}}\underset{\approx}{\boldsymbol{A}})^L \boldsymbol{:} \underset{\approx}{\boldsymbol{X}} = 2(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^{ML} \boldsymbol{:} \underset{\approx}{\boldsymbol{X}} \tag{17.81}$$

Note that

$$\underset{\approx}{\boldsymbol{Y}}^T = \underset{\approx}{\boldsymbol{B}} \bullet \underset{\approx}{\boldsymbol{X}}^T \bullet \underset{\approx}{\boldsymbol{A}}^T + \underset{\approx}{\boldsymbol{A}} \bullet \underset{\approx}{\boldsymbol{X}}^T \bullet \underset{\approx}{\boldsymbol{B}}^T = (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}} + \underset{\approx}{\boldsymbol{B}}\underset{\approx}{\boldsymbol{A}})^L \boldsymbol{:} \underset{\approx}{\boldsymbol{X}}^T = 2(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}})^{ML} \boldsymbol{:} \underset{\approx}{\boldsymbol{X}}^T \tag{17.82}$$

Therefore

$$\operatorname{sym}\underset{\approx}{\boldsymbol{Y}} = (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}} + \underset{\approx}{\boldsymbol{B}}\underset{\approx}{\boldsymbol{A}})^L \boldsymbol{:} (\operatorname{sym}\underset{\approx}{\boldsymbol{X}}) \tag{17.83}$$

Equivalently,

$$\operatorname{sym}\underset{\approx}{\boldsymbol{Y}} = (\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}} + \underset{\approx}{\boldsymbol{B}}\underset{\approx}{\boldsymbol{A}})^\lambda \boldsymbol{:} \underset{\approx}{\boldsymbol{X}}. \tag{17.84}$$

Note that

$$(\underset{\approx}{\boldsymbol{A}}\underset{\approx}{\boldsymbol{B}} + \underset{\approx}{\boldsymbol{B}}\underset{\approx}{\boldsymbol{A}})^\lambda = \underset{\approx}{\boldsymbol{A}} \wedge \underset{\approx}{\boldsymbol{B}} \text{ if both } \underset{\approx}{\boldsymbol{A}} \text{ and } \underset{\approx}{\boldsymbol{B}} \text{ are symmetric.} \tag{17.85}$$

18. Cartesian coordinate/basis transformations

Introduction to a general change of basis

Consider a simple array:

$$\{u\} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{18.1}$$

We can expand $\{u\}$ as a linear combination

$$\{u\} = \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{18.2}$$

In this expansion, a and b are components with respect to the basis arrays $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can double the length of the first base array if we half its coefficient:

$$\{u\} = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{a}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{18.3}$$

Here, $a/2$ and b are components with respect to the different basis arrays $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Changing a basis requires components to change in a particular way so that *the sum of components times basis vectors never changes!* Here, doubling the basis vector required halving the component. Components change in a manner opposing or counter-balancing the change in basis. For this reason, the components are called "**contravariant**".

If a vector observer-free, that does not imply that it will look the same to differently positioned or differently oriented observers. *REB: explain what it does mean!*

23. Material and Tensor symmetry

Theory of material symmetry is covered in a rigorous and elegant manner through the use of group theory. Here will only give a simple overview of the results that are immediately accessible to engineers who lack a background in this branch of mathematics. The symmetry of a material is measured by how its properties under load vary with respect to changes in the material's initial (unloaded) material orientation. If the material properties are unaffected by the initial material orientation, then the material is said to be **proper isotropic**. If the material properties are additionally unchanged upon a *reflection*, then the tensor is **strictly isotropic**. If the material properties are unaffected by rotation about some given vector $\underset{\sim}{a}$, as for unidirectional fiber-reinforced plastics or idealized plywood, then the material is **transversely isotropic**. If the material properties are unaffected by $90°$ rotations about a specified set of orthonormal reference vectors, then the material has **cubic symmetry**.

Material symmetry does not imply tensor symmetry. A separate concept, *tensor* symmetry, refers to whether or not components of a tensor are invariant under orthogonal transformations. Material symmetry does not necessarily imply the same symmetry in material property tensors. An isotropic material, for example, does *not* generally have an isotropic stiffness. To understand this statement, suppose you squash a rubber ball made of an isotropic material. Material isotropy means that the amount of force required the same for all squashing directions. You can squash horizontally, vertically or any other direction — the force magnitude will be the same and the induced stress field relative to the squashing direction will be the same. However, the material stiffness tensor quantifies the *increment* in force required to obtain an *increment* in deformation. Pushing or pulling an isotropic *material* in a given direction will cause the stiffness *tensor* to become transversely isotropic about that direction. Getting an additional increment of deformation might be harder in the squashing direction than in the relatively undeformed direction perpendicular to the squashing direction. An isotropic strain *increment* applied to a distorted isotropic material usually requires an anisotropic stress *increment*. This is called **induced anisotropy**. Improper accounting of induced anisotropy is probably the main contributor to non-predictiveness of material constitutive models to non-monotonic loading paths.

In general, a hyperelastic material is one for which there exists a potential function of strain $\phi(\underset{\sim}{\varepsilon})$ such that the work conjugate stress $\underset{\sim}{\sigma}$ and the corresponding stiffness tensor $\underset{\approx}{C}$ are determined by

$$\underset{\sim}{\sigma} = \frac{d\phi(\underset{\sim}{\varepsilon})}{d\underset{\sim}{\varepsilon}} \qquad\qquad \underset{\approx}{C} = \frac{d\underset{\sim}{\sigma}}{d\underset{\sim}{\varepsilon}} = \frac{d^2\phi}{d\underset{\sim}{\varepsilon}\, d\underset{\sim}{\varepsilon}} \tag{23.1}$$

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

If the *material* is isotropic, then the potential function depends on strain only through its invariants. Using the chain rule twice to compute the second-derivative of the potential function therefore produces a stiffness that is anisotropic [see Eq. (23.85)]. Under distortion (*i.e.*, when the strain deviator is nonzero and non-infinitesimal), the stiffness can be equated to an isotropic tensor only in the exceptional case that the bulk modulus depends only on volumetric strain and the shear modulus is *constant*. As a practicing engineer, you should feel free to tentatively presume that the stiffness is isotropic. However, if your analysis of data leads you to conclude that the shear modulus varies, then you must reject your premise of an isotropic stiffness and re-analyze the data allowing for distortion-induced anisotropy [39].

What is tensor isotropy?

There are two common ways to define tensor isotropy.

(i) Definition 1: a tensor is *strictly* isotropic if its components are unchanged upon any orthonormal change in basis

(ii) Definition 2: a tensor is *proper*-isotropic if its components are unchanged upon any *same-handed* change in basis

These definitions apply to any order tensor. Consider a second-order tensor $\underset{\approx}{A}$ of type V_3^2. According to Eq. (18.49) tensor components change under a change in basis, but we now seek restrictions on those components such that they *don't* change. Applying the above definitions of isotropy,

Strict isotropy means $Q_{ip}Q_{jq}A_{pq} = A_{ij}$ for any orthogonal matrix $[Q]$. (23.2)

Proper isotropy means $R_{ip}R_{jq}A_{pq} = A_{ij}$ for any proper orthogonal matrix $[R]$. (23.3)

In terms of the Rayleigh product, defined on page 1022, these conditions may be written in direct notation as

$$\textbf{Strict isotropy means } \underset{\approx}{Q} > \underset{\approx}{A} = \underset{\approx}{A} \qquad \forall \text{ orthogonal } \underset{\approx}{Q}. \tag{23.4}$$

$$\textbf{Proper isotropy means } \underset{\approx}{R} > \underset{\approx}{A} = \underset{\approx}{A} \qquad \forall \text{ proper orthogonal } \underset{\approx}{R}. \tag{23.5}$$

A proper orthogonal tensor (i.e., an orthogonal tensor with determinant $+1$) is a rotation operation. When a tensor is proper-isotropic, it "looks the same" no matter what orientation you view it from. Likewise, if you hold yourself fixed, then the tensor "looks the same" no matter how you "turn" it. The geometrical analog would be a sphere or something with spherical symmetry (like a hollow spherical shell) that looks the same from all perspectives. There is no guarantee that a proper-isotropic tensor won't look different if you invert it (i.e., if you switch to a left-handed basis). Strict isotropy insists that the tensor must also look the same for both rotations and reflections.

Proper-isotropy is a "weaker" condition* because satisfaction of Eq. 23.2 automatically guarantees satisfaction of 23.3, but not vice-versa. Which definition is more useful? In the vast majority of physics applications, you want to know when something will be unchanged upon a rotation, but you don't care what happens upon a (usually non-physical) reflection. Knowledge of proper-isotropy (even when strict-isotropy does not hold) is very useful and should be tested first. Thus, unless otherwise stated, we take the term "isotropic" to mean "proper-isotropic."

As a general rule, to determine the most general form for an isotropic tensor, you should first consider restrictions placed on the components of a tensor for *particular choices* of the rotation tensor, which will simplify your subsequent analysis exploring the set of all possible rotations. Good "particular" choices for the rotations are $90°$ rotations about the coordinate axes:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{23.6}$$

The component restrictions arising from these special choices will give you *necessary* conditions for isotropy. Frequently, these necessary conditions turn out to also be sufficient conditions, but you have to prove it.

To deduce the most general form for an isotropic *vector* $\underset{\sim}{v}$, you would demand that its components satisfy the equation $R_{ip}v_p = v_i$ for all proper rotations. In matrix form,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{23.7}$$

This must hold for all proper-orthogonal matrices $[R]$. Consequently, it must hold for any of the special cases in Eq. 23.6. Considering the first case,

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} -v_2 \\ v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{23.8}$$

* Statement A is said to be "weaker" than statement B if B implies A. For example, $x \le 0$ is weaker than $x < 0$.

With this simple test, we have already learned an important *necessary* condition for a vector to be isotropic. Namely, v_2 must equal $-v_1$, and v_2 must also equal v_1. The only way one number can equal another number *and the negative* of that other number is if both numbers are zero. Thus, an isotropic vector would *necessarily* have the form $<0, 0, v_3>$. This form is necessary, but it is not sufficient for the vector to be isotropic. Using the second choice in Eq. 23.6 along with the new knowledge that an isotropic vector must be of the form $<0, 0, v_3>$ gives:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} v_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}. \tag{23.9}$$

This result tells us that v_3 itself must be zero. In other words, a necessary requirement for a vector to be isotropic is that the vector's components must *all* be zero. This is also *sufficient* because then Eq. 23.7 is then satisfied for *all* rotations (not just the special ones) when the vector is zero. Thus, the zero vector is the *only* isotropic vector.

Isotropy of tensors. Look now at second-order tensors. If $\underset{\approx}{A}$ and $\underset{\approx}{B}$ (each of type V_3^2) are isotropic, then

$$R_{ip}R_{jq}A_{pq} = A_{ij} \qquad \text{and} \qquad R_{ip}R_{jq}B_{pq} = B_{ij}. \tag{23.10}$$

It follows that any linear combination $\alpha\underset{\approx}{A} + \beta\underset{\approx}{B}$ will be isotropic because

$$R_{ip}R_{jq}(\alpha A_{pq} + \beta B_{pq}) = \alpha(R_{ip}R_{jq}A_{pq}) + \beta(R_{ip}R_{jq}B_{pq}) = \alpha A_{ij} + \beta B_{ij}. \tag{23.11}$$

Important consequence. Since any linear combination of isotropic tensors (of a given V_m^n type) will itself be isotropic, it follows that the set of isotropic tensors (of that type) is a linear manifold. Thus, the zero tensor will always be an isotropic tensor. More importantly, if there exist *any* non-trivial (i.e., nonzero) isotropic tensors, then there must exist a basis for the subspace of *all* isotropic tensors of that type. We will soon prove that a second-order tensor of type V_3^2 (i.e., a second-order tensor in a 3D space) is isotropic if and only if it is some multiple of the identity tensor. Consequently, the identity tensor itself is a basis for the subspace of isotropic tensors of type V_3^2. In this case, there's only one basis tensor, so this must be a one-dimensional space. Projecting an arbitrary tensor onto this space (using the projection techniques covered elsewhere in this book) gives you the *isotropic part* of that tensor. Projecting perpendicular to this space gives you the deviatoric part.

Proper isotropic tensors of type V_3^2 are always multiples of the identity tensor, but the same is not true for tensors of type V_2^2 (i.e., second-order tensors in a 2D space). For V_2^2, the space of proper isotropic tensors is two-dimensional: one basis tensor is the identity, but there is also a second basis tensor, discussed below. In general, for tensors of type V_m^n, the space of isotropic tensors depends on both the dimension of the space, m, and the order of the tensor, n.

Isotropic second-order tensors in 3D space

Referring to Eq. (23.3), a tensor of type V_3^2 (i.e., a second-order tensor in 3D space) is proper-isotropic if and only if

$$[T] = [R][T][R]^T \text{ for all proper-orthogonal } [R]. \tag{23.12}$$

By applying this condition using the special cases in Eq. (23.6), you will find that a necessary condition for proper isotropy is that $[T]$ must be a multiple of the identity. This condition is also *sufficient* because then Eq. (23.3) is then satisfied for any rotation R. Moreover, this condition is sufficient for *strict* isotropy because, if the tensor is a multiple of the identity, then Eq. (23.2) is satisfied for all orthogonal $[Q]$. Thus, components of a V_3^2 isotropic tensor are always of the form

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \tag{23.13}$$

where α is an arbitrary scalar. Isotropic second-order tensors in 3D space therefore span a one-dimensional space since only one scalar is needed. The identity tensor $\underset{\approx}{\boldsymbol{I}}$ is a basis for this space. Any general tensor $\underset{\approx}{\boldsymbol{B}}$ may be projected to its isotropic part by the operation

$$\text{iso}\underset{\approx}{\boldsymbol{B}} = \frac{\underset{\approx}{\boldsymbol{I}}(\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{B}})}{\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{I}}}. \tag{23.14}$$

Note that $\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{B}} = \text{tr}\underset{\approx}{\boldsymbol{B}}$ and $\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{I}} = \text{tr}\underset{\approx}{\boldsymbol{I}} = 3$. Hence,

$$\text{iso}\underset{\approx}{\boldsymbol{B}} = \frac{1}{3}(\text{tr}\underset{\approx}{\boldsymbol{B}})\underset{\approx}{\boldsymbol{I}}. \tag{23.15}$$

This is a familiar result to anyone who has taken Continuum Mechanics. The idea of finding the isotropic part by projecting to the space of isotropic tensors becomes less obvious when considering tensors in spaces of different dimensions.

A scalar measure of isotropy. In engineering applications, one may encounter a tensor that is nearly — but not quite — isotropic. How do you quantify "degree of isotropy?" This question is similar to asking "to what extent does a vector point East?" Most people would use the angle θ formed between the vector and East. Alternatively, to obtain a scaled "fuzzy logical" that equals zero if the vector points north or south, –1 if it points west, and +1 if it points exactly east, you could define "Eastliness" of a vector by $1 - \theta/(\pi/2)$. A scalar measure of *tensor* isotropy can be defined similarly. In this case, we seek the extent of alignment of a tensor $\underset{\approx}{\boldsymbol{B}}$ with the identity tensor $\underset{\approx}{\boldsymbol{I}}$, where $\underset{\approx}{\boldsymbol{I}}$ is regarded as defining the direction "East." The angle between $\underset{\approx}{\boldsymbol{B}}$ and $\underset{\approx}{\boldsymbol{I}}$ is computed using Eq. (6.39). The "East" component of $\underset{\approx}{\boldsymbol{B}}$ is its inner-product with $\underset{\approx}{\boldsymbol{I}}/\sqrt{3}$ (namely, $\text{tr}\underset{\approx}{\boldsymbol{B}}/\sqrt{3}$), where $\sqrt{3}$ simply normalizes the identity tensor (i.e., like using a *unit* vector parallel to the [111] direction to find the projection of a pseudo stress vector onto the hydrostat). The remainder of the second-order tensor is its deviatoric part. Using an ArcTangent in the range from 0 to π, a scalar measure of isotropy of a second-order tensor is

$$1 - \frac{2}{\pi}\mathrm{ArcTan}\left(\frac{\|\mathrm{dev}\underset{\approx}{\boldsymbol{B}}\|}{\mathrm{tr}\underset{\approx}{\boldsymbol{B}}/\sqrt{3}}\right) \qquad \text{scalar measure of second-order tensor isotropy} \qquad (23.16)$$

Equivalently,

$$1 - \frac{2}{\pi}\mathrm{ArcCos}\left(\frac{\mathrm{tr}\underset{\approx}{\boldsymbol{B}}/\sqrt{3}}{\|\underset{\approx}{\boldsymbol{B}}\|}\right) \qquad \text{scalar measure of second-order tensor isotropy} \qquad (23.17)$$

Both of these expressions give the same result and can be interpreted geometrically as illustrated in Fig. 23.1. The set of isotropic tensors is one-dimensional (all are multiples of the identity). Because the inner product of an isotropic tensor with *any* deviatoric tensor is zero, the set of deviatoric tensors is an 8-D hyperplane (illustrated as a disk in the figure perpendicular to the identity tensor). If attention is restricted to symmetric tensors (which have 6 independent components), then the deviatoric plane is 5-D (because 6-1=5).

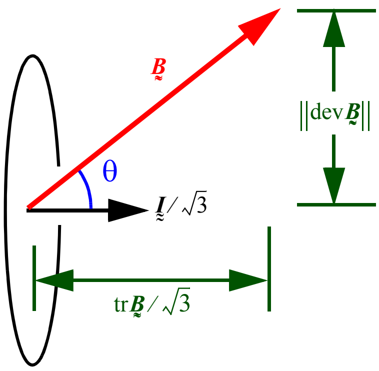

Figure 23.1. Quantifying the extent of isotropy of a tensor via degree of alignment with the identity tensor.

The multiplier of $2/\pi$ in Eq. (23.17) simply normalizes the angle θ between $\underset{\approx}{\boldsymbol{B}}$ and $\underset{\approx}{\boldsymbol{I}}$ so that the scalar measure of isotropy ranges from -1 when the tensor is isotropic and aligned with $-\underset{\approx}{\boldsymbol{I}}$, to 0 when the tensor is purely deviatoric, and finally $+1$ when the tensor is isotropic and aligned with $\underset{\approx}{\boldsymbol{I}}$.

Recognizing that the key feature of anisotropy is the fact that components change in response to superimposed rotation, Rychlewski [79] defined the **orbit** of a tensor to be the maximum distanced between any two tensors attainable by rotation:

$$\mathrm{orbit}(\underset{\approx}{\boldsymbol{B}}) = \max_{\underset{\approx}{\boldsymbol{Q}}\in\mathrm{orth}} \left\|\underset{\approx}{\boldsymbol{Q}} \bullet \underset{\approx}{\boldsymbol{B}} \bullet \underset{\approx}{\boldsymbol{Q}}^T - \underset{\approx}{\boldsymbol{B}}\right\| \qquad (23.18)$$

where “orth” stands for the set of all orthogonal tensors. With this, Rychlewski proposed a nondimensionalized scalar measure of anisotropy as

$$\frac{\mathrm{orbit}(\underset{\approx}{\boldsymbol{B}})}{2\|\underset{\approx}{\boldsymbol{B}}\|} \qquad (23.19)$$

The scalar measures of isotropy that were introduced above for second-order tensors are generalized to higher-order tensors in subsequent sections. Alternative scalar measures of tensor isotropy (or anisotropy) have been considered in a number of contexts and disciplines [72, 5, 35, 79], with some of these works also generalizing the quantification of anisotropy to higher-order contexts. In all cases, a scalar measure of isotropy is just an alternative invariant of the tensor. The only differences are the usefulness of these invariants to quantify anisotropy for a particular applications.

Isotropic second-order tensors in 2D space

To investigate isotropy of tensors of type V_2^2 (i.e., tensors in two-dimensions whose components can specified using 2×2 matrices), first we need to identify the general form for an orthogonal tensor in this space.

$$\text{Let } Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{23.20}$$

We seek restrictions on the components $(a, b, c, \text{and } d)$ such that

$$[Q]^T[Q] = [I], \tag{23.21}$$

or

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{23.22}$$

Multiplying this out shows that the components must satisfy

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + dc &= 0. \end{aligned} \tag{23.23}$$

We can satisfy the first two constraints automatically by setting

$$\begin{aligned} a &= \cos\alpha, \ c = \sin\alpha \\ b &= \cos\beta, \ d = \sin\beta. \end{aligned} \tag{23.24}$$

Satisfying the last constraint requires

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = 0, \tag{23.25}$$

or

$$\cos(\beta - \alpha) = 0, \tag{23.26}$$

or

$$\beta = \alpha \pm \frac{\pi}{2}. \tag{23.27}$$

Putting this back into Eq. 23.24 with the choice $\beta = \alpha + \frac{\pi}{2}$ gives

$$\begin{aligned} a &= \cos\alpha, \ c = \sin\alpha \\ b &= -\sin\alpha, \ d = \cos\alpha. \end{aligned} \tag{23.28}$$

Putting this result back into Eq. 23.20 yields a proper orthogonal matrix, so we will denote it by $[R]$. Namely,

$$[R] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{23.29}$$

On the other hand, substituting Eq. 23.27 back into Eq. 23.24 using the alternative choice $\beta = \alpha - \frac{\pi}{2}$ gives

$$\begin{aligned} a &= \cos\alpha, \; c = \sin\alpha \\ b &= \sin\alpha, \; d = -\cos\alpha. \end{aligned} \tag{23.30}$$

which, using Eq. (23.20), yields an improper orthogonal matrix,

$$[Q] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}. \tag{23.31}$$

Equation (23.29) is the most general form for a proper orthogonal matrix in 2D and Eq. (23.31) is the most general form for an improper matrix.

We are now ready to explore the nature of isotropic tensors in 2D. For a second order tensor to be proper isotropic, its components must satisfy

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{23.32}$$

Considering, as a special case, $\alpha = \pi/2$, this equation becomes

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{23.33}$$

or

$$A_{11} = A_{22} \quad \text{and} \quad A_{12} = -A_{21}. \tag{23.34}$$

Since this result was obtained by considering a special rotation, we only know it a *necessary* condition for isotropy. However, substituting this condition back into Eq. 23.32 shows that it is also *sufficient*. Consequently, the most general form for an isotropic tensor referenced to 2D space is of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \tag{23.35}$$

where a and b are arbitrary parameters. Any tensor in 2D space that is of this proper-isotropic form may be expressed as a linear combination of the following two primitive basis tensors:

$$[\underset{\approx}{I}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [\underset{\approx}{\varepsilon}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{23.36}$$

Note that $\underset{\approx}{\varepsilon}$ is the 2D version of the permutation symbol; namely, ε_{ij} is zero if $i=j$, it is +1 if $ij=12$, and it is –1 if $ij=21$. In two dimensions, the (proper) isotropic part of a second-order tensor $\underset{\approx}{F}$ would be obtained by projecting the tensor onto the space spanned by the basis in Eq. (23.36). This basis is orthogonal, but not normalized, so the appropriate projection operation is

$$\begin{aligned}\mathrm{iso}\underset{\approx}{F} &= \frac{\underset{\approx}{I}(\underset{\approx}{I}:\underset{\approx}{F})}{\underset{\approx}{I}:\underset{\approx}{I}} + \frac{\underset{\approx}{\varepsilon}(\underset{\approx}{\varepsilon}:\underset{\approx}{F})}{\underset{\approx}{\varepsilon}:\underset{\approx}{\varepsilon}} \\ &= \frac{1}{2}\underset{\approx}{I}(\underset{\approx}{I}:\underset{\approx}{F}) + \frac{1}{2}\underset{\approx}{\varepsilon}(\underset{\approx}{\varepsilon}:\underset{\approx}{F}) \\ &= \frac{\underset{\approx}{I}(F_{11}+F_{22})}{2} + \frac{\underset{\approx}{\varepsilon}(F_{12}-F_{21})}{2}\,. \end{aligned} \tag{23.37}$$

In component form,

$$[\mathrm{iso}\underset{\approx}{F}] = \frac{1}{2}\begin{bmatrix} F_{11}+F_{22} & F_{12}-F_{21} \\ F_{21}-F_{12} & F_{11}+F_{22} \end{bmatrix}. \tag{23.38}$$

Incidentally, the 2D polar rotation tensor associated with any 2D $\underset{\approx}{F}$ tensor is proportional to $\mathrm{iso}\underset{\approx}{F}$. The proportionality factor is found by requiring the determinant of the polar rotation to equal unity. Specifically,

$$[\underset{\approx}{R}] = \frac{1}{\sqrt{(F_{11}+F_{22})^2+(F_{12}-F_{21})^2}}\begin{bmatrix} F_{11}+F_{22} & F_{12}-F_{21} \\ F_{21}-F_{12} & F_{11}+F_{22} \end{bmatrix} \qquad \text{for 2D space only!} \tag{23.39}$$

This formula provides a particularly convenient (and mysteriously not well known) method for finding the polar decomposition for 2D problems.

Recall that proper isotropy is necessary — *but not always sufficient* — for strict-isotropy. To be strict isotropic, the tensor's components must also be unchanged under reflections (i.e., using the matrix $[Q]$ from Eq. 23.31). Namely

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \tag{23.40}$$

Considering $\alpha = 0$ shows that b must necessarily be zero, which is also sufficient to satisfy Eq. 23.40 for all orthogonal tensors. Consequently, a *strictly* isotropic second-order tensor in 2D must be a multiple of the identity. The 2D permutation tensor $\underset{\approx}{\varepsilon}$ is proper-isotropic, but not strictly isotropic (its components will change sign upon changing to a differently handed basis). In dyadic form, the permutation tensor may be written

$$\underset{\approx}{\varepsilon} = \underset{\sim}{e}_1\underset{\sim}{e}_2 - \underset{\sim}{e}_2\underset{\sim}{e}_1\,. \tag{23.41}$$

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Changing to a different-handed basis (i.e., exchanging $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$) automatically changes the sign of the components. Therefore, if you define the 2D permutation tensor by Eq. (23.41), then you do not need any special treatment (such as changing sign of the permutation symbol) when switching handedness of the basis. The permutation *symbol* remains unchanged, while components of the permutation tensor will automatically change sign in a left-handed basis.

Isotropic fourth-order tensors in 3D space

Turning our attention back to tensors in 3D space, a *fourth-order* tensor is proper isotropic if its components satisfy

$$R_{ip}R_{jq}R_{kr}R_{ls}c_{pqrs} = c_{ijkl} \tag{23.42}$$

The process for finding constraints on the components is similar to what was done earlier for lower-order tensors. You can fire up your favorite symbolic math program and apply special case rotations such as those in Eq. (23.6) as well as some additional ones such as $45°$ rotations until you have produced enough necessary conditions to also be sufficient to satisfy Eq. (23.42). Alternatively, you can look up the proof in just about any tensors textbook, such as Ref. [4]. In the end, you will find that the most general form for an isotropic fourth-order tensor is

$$c_{ijrs} = \alpha(\delta_{ij}\delta_{rs}) + \beta(\delta_{ir}\delta_{js}) + \gamma(\delta_{is}\delta_{jr}),\ \text{for some scalars } \alpha, \beta, \gamma \tag{23.43}$$

This expression has three arbitrary parameters, so the space of isotropic fourth-order tensors (of type V_3^4) is three-dimensional.* Using the leafing and palming operators defined in Eqs. (17.49) and (17.54), the most general direct-notation form of an isotropic fourth-order tensor is†

$$\underset{\approx\approx}{c} = \alpha(\underset{\approx}{I}\underset{\approx}{I}) + \beta(\underset{\approx}{I}\underset{\approx}{I})^{\mathrm{L}} + \gamma(\underset{\approx}{I}\underset{\approx}{I})^{\Gamma} \tag{23.44}$$

The functional meaning of this tensor is explored by considering how it affects an arbitrary second-order tensor $\underset{\approx}{X}$. Namely,

$$\underset{\approx\approx}{c}:\underset{\approx}{X} = \alpha(\mathrm{tr}\underset{\approx}{X})\underset{\approx}{I} + \beta\underset{\approx}{X} + \gamma\underset{\approx}{X}^T \tag{23.45}$$

In other words, this result produces a linear combination of the tensor $\underset{\approx}{X}$, its transpose $\underset{\approx}{X}^T$, and the identity $\underset{\approx}{I}$. Alternatively, if the above equation holds for any choice of $\underset{\approx}{X}$, then you may conclude that $\underset{\approx\approx}{c}$ is isotropic. In other words, the tensor $\underset{\approx\approx}{c}$ is isotropic if and only if its output when acting on an arbitrary second-order tensor $\underset{\approx}{X}$ is a fixed linear combination of $\underset{\approx}{I}$, $\underset{\approx}{X}$, and $\underset{\approx}{X}^T$. Introducing the decomposition $\underset{\approx}{X} = \mathrm{iso}\underset{\approx}{X} + \mathrm{symdev}\underset{\approx}{X} + \mathrm{skw}\underset{\approx}{X}$, it follows

* The space is two dimensional if one imposes a minor symmetry restriction that $c_{ijrs}=c_{jirs}=c_{ijsr}$. That's why isotropic elastic stiffness tensors have only two independent stiffness moduli.

† As explained later in this section, other *equally general* forms exist, so don't stop reading here — we recommend the more useful expansions yet to come!

that $\underset{\underset{\approx}{\approx}}{\boldsymbol{c}}$ is isotropic if and only if its output when acting on an arbitrary second-order tensor $\underset{\approx}{\boldsymbol{X}}$ is a fixed linear combination of $\text{iso}\underset{\approx}{\boldsymbol{X}}$, $\text{symdev}\underset{\approx}{\boldsymbol{X}}$, and $\text{skw}\underset{\approx}{\boldsymbol{X}}$. This observation motivates our recognizing that the three basis tensors in the parentheses of Eq. (23.43) are only one choice for the basis for the set of fourth-order isotropic tensors. In materials modelling, a more convenient basis for the space of isotropic fourth-order tensors is

$$P^{\text{iso}}_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl} \tag{23.46a}$$

$$P^{\text{symdev}}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl} \tag{23.46b}$$

$$P^{\text{skew}}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,. \tag{23.46c}$$

Thus, in direct symbolic notation, a fourth-order tensor $\underset{\underset{\approx}{\approx}}{\boldsymbol{c}}$ is isotropic if and only if it is expressible in the form

$$\underset{\underset{\approx}{\approx}}{\boldsymbol{c}} = a\underset{\underset{\approx}{\approx}}{\boldsymbol{P}}^{\text{iso}} + b\underset{\underset{\approx}{\approx}}{\boldsymbol{P}}^{\text{symdev}} + c\underset{\underset{\approx}{\approx}}{\boldsymbol{P}}^{\text{skew}} \quad \text{for some scalars} \quad a,b,c. \tag{23.47}$$

This result is called the spectral **representation theorem** for fourth-order isotropic tensors [*cf.*58], and it implies that, for any tensor $\underset{\approx}{\boldsymbol{X}}$,

$$\underset{\underset{\approx}{\approx}}{\boldsymbol{c}}:\underset{\approx}{\boldsymbol{X}} = a(\text{iso}\underset{\approx}{\boldsymbol{X}}) + b(\text{symdev}\underset{\approx}{\boldsymbol{X}}) + c(\text{skw}\underset{\approx}{\boldsymbol{X}}) \tag{23.48}$$

Conversely, if this relationship holds *for arbitrary* $\underset{\approx}{\boldsymbol{X}}$, you may conclude that the tensor $\underset{\underset{\approx}{\approx}}{\boldsymbol{c}}$ is isotropic.

Exercise 23.1 Prove that the representations in Eqs. (23.44) and (23.47) truly are related by a simple change of basis by showing that the coefficients are related linearly by

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}1/3 & 1/3 & 0\\ 0 & 1/2 & 1/2\\ 0 & 1/2 & -1/2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

Hint: The matrix is a basis transformation operator. You can get it in a variety of ways, such as expressing the fourth-order tensors in Eq. (23.47) in terms of those in (23.44), but that approach isn't obvious if you don't already know those relationships. An alternative approach expresses the tensors on the right side of Eq. (23.48) in terms of those in (23.45). Then just match coefficients of like terms.

Mandel matrices for isotropic fourth-order basis tensors. Using the algorithm on page 354, the 9×9 Mandel matrices for the fourth-order basis tensors appearing in Eq. (23.44) are

Add these (23.49)

Similarly, the Mandel matrices for the alternative isotropic basis in Eq. (23.47) are

Add these (find them elsewhere already?) (23.50)

For small-strain classical (non-Cosserat) elasticity, the minor-symmetric isotropic elastic **stiffness** is a special case of Eq. (23.47) for which $a = 3K$, $b = 2G$, and $c = 0$:

$$\underset{\approx}{\boldsymbol{C}} = (3K)\underset{\approx}{\boldsymbol{P}}^{\text{iso}} + (2G)\underset{\approx}{\boldsymbol{P}}^{\text{symdev}}. \tag{23.51}$$

Here K is the bulk modulus and G is the shear modulus. Symmetry of stress and strain require the stiffness to be minor symmetric, which is why there is no term involving $\underset{\approx}{\boldsymbol{P}}^{\text{skew}}$. Actually, the term is still there, but its coefficient is zero.

Even though the above expansion involves fourth-order tensors that are more complicated than those in Eq. (23.43), this expansion is attractive because it is the *spectral* expansion, where $3K$ and $2G$ are eigenvalues of the stiffness and the P-tensors are the corresponding eigenprojectors. Having the spectral form allows you to easily invert the stiffness to obtain **compliance*** as

$$\underset{\approx}{\boldsymbol{S}} = \underset{\approx}{\boldsymbol{C}}^{-1} = \left(\frac{1}{3K}\right)\underset{\approx}{\boldsymbol{P}}^{\text{iso}} + \left(\frac{1}{2G}\right)\underset{\approx}{\boldsymbol{P}}^{\text{symdev}}. \tag{23.52}$$

The Mandel matrix form for the stiffness $\underset{\approx}{\boldsymbol{C}}$ and compliance $\underset{\approx}{\boldsymbol{S}}$ are, respectively,

Add stiffness matrix (probably move from elsewhere already in the document) Express it in terms of the constrained modulus and Lame modulus (23.53)

Add compliance matrix (probably move from elsewhere already in the document) Express it in terms of the Young's modulus and Poisson's ratio (23.54)

where

$$C := K + \frac{4}{3}G \qquad \text{(called the \textbf{constrained modulus})} \tag{23.55}$$

$$\lambda := K - \frac{2}{3}G \qquad \text{(called the \textbf{Lame modulus})} \tag{23.56}$$

$E :=$ *Add this* (called **Young's modulus**) (23.57)

$\nu :=$ *add this* (called **Poisson's ratio**) (23.58)

The fourth-order tensors in Eq. (23.46) are all constructed from linear combinations of the primitive basis in Eq. (23.43). Even though the component formulas for this alternative basis are considerably more complicated, the properties of this basis are wonderful. Specifically, the basis of Eq. (23.46) comprises *complementary projectors*! By this we mean

$$\boxed{P^{\text{iso}}_{ijkl}P^{\text{iso}}_{klmn} = P^{\text{iso}}_{ijmn}},\quad P^{\text{iso}}_{ijkl}P^{\text{symdev}}_{klmn} = 0,\quad P^{\text{iso}}_{ijkl}P^{\text{skew}}_{klmn} = 0 \tag{23.59}$$

$$P^{\text{symdev}}_{ijkl}P^{\text{iso}}_{klmn} = 0^{\text{iso}},\quad \boxed{P^{\text{symdev}}_{ijkl}P^{\text{symdev}}_{klmn} = P^{\text{symdev}}_{ijmn}},\quad P^{\text{symdev}}_{ijkl}P^{\text{skew}}_{klmn} = 0 \tag{23.60}$$

$$P^{\text{skew}}_{ijkl}P^{\text{iso}}_{klmn} = 0,\quad P^{\text{skew}}_{ijkl}P^{\text{symdev}}_{klmn} = 0,\quad \boxed{P^{\text{skew}}_{ijkl}P^{\text{skew}}_{klmn} = P^{\text{skew}}_{ijmn}} \tag{23.61}$$

* It is conventional to use "C" for stiffness and "S" for compliance. Go figure!

TENSOR ALGEBRA

Recall that second-order tensors don't really take on any meaning until they act on a vector. Likewise, the meaning of a fourth-order tensor should be inferred by what it does when it acts on a second-order tensor. For any tensor $\underset{\approx}{B}$, note that

$$P^{\text{iso}}_{ijkl} B_{kl} = \frac{1}{3} B_{kk} \delta_{ij} \tag{23.62a}$$

$$P^{\text{symdev}}_{ijkl} = \frac{1}{2}(B_{ij} + B_{ji}) - \frac{1}{3} B_{kk} \delta_{ij} \tag{23.62b}$$

$$P^{\text{skew}}_{ijkl} B_{kl} = \frac{1}{2}(B_{ij} - B_{ji}) . \tag{23.62c}$$

Thus, P^{iso}_{ijkl} returns the isotropic part of $\underset{\approx}{B}$, P^{symdev}_{ijkl} returns the symmetric-deviatoric part of $\underset{\approx}{B}$, and P^{skew}_{ijkl} returns the skew part of $\underset{\approx}{B}$. The superscript labels such as "symdev" refer to what the fourth-order tensor does to second-order tensors. The fourth-order tensor P^{symdev}_{ijkl} is isotropic despite the appearance of "dev" in its name.

Hooke's law: spectral form

> Exercise 23.2 In elasticity, the details of the stiffness tensor are often not of as much interest as the end result of applying the stiffness to a strain $\underset{\sim}{\varepsilon}$ to evaluate stress $\underset{\sim}{\sigma}$. If stiffness is given by the formula in Eq. (23.51), show that the linear elasticity formula $\underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon}$ leads to the following results:
>
> $$p = K\varepsilon_v, \quad \text{where } p = \frac{1}{3}\text{tr}\underset{\sim}{\sigma} \quad \text{and } \varepsilon_v = \text{tr}\underset{\sim}{\varepsilon} \tag{23.63a}$$
>
> $$\underset{\sim}{S} = 2G\underset{\sim}{\gamma}, \quad \text{where } \underset{\sim}{S} = \text{dev}\underset{\sim}{\sigma} \quad \text{and } \underset{\sim}{\gamma} = \text{dev}(\underset{\sim}{\varepsilon}) \tag{23.63b}$$

Hint: Use the physical meaning of the projectors, as describe in the paragraph after Eq. (23.62). Then separately take the isotropic and deviatoric parts of the result. Caution: the tensor $\underset{\sim}{S}$ *with two under-tildes is the stress deviator, not to be confused with the compliance* $\underset{\approx}{S}$*, which has four under-tildes.*

Hooke's law: Lamé form

> Exercise 23.3 In the Lamé formulation of Hooke's law, stress $\underset{\sim}{\sigma}$ depends on strain $\underset{\sim}{\varepsilon}$ according to $\underset{\sim}{\sigma} = 2\mu\underset{\sim}{\varepsilon} + \lambda(\text{tr}\underset{\sim}{\varepsilon})\underset{\sim}{I}$, where μ and λ are constants called the Lamé moduli.
>
> (a) Prove that this formula can be written in the form $\underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon}$ where $\underset{\approx}{C}$ is isotropic. Specifically express C_{ijkl} as a linear combination of $\delta_{ij}\delta_{rs}$, $\delta_{ir}\delta_{js}$, and $\delta_{is}\delta_{jr}$.
> (b) Demonstrate equivalence of the Lamé and spectral forms of Hooke's law by deriving specific formulas expressing K and G in Eq. (23.63) as functions of μ and λ.

Hint: (a) Do this by using the representation theorem to note that $\underset{\sim}{\varepsilon}$ *and* $(\text{tr}\underset{\sim}{\varepsilon})\underset{\sim}{I}$ *are each linear with respect to* $\underset{\sim}{\varepsilon}$*. Find the associated tensors using techniques similar to those discussed on page 395 for second-order tensors. Be sure to impose minor symmetry since stress and strain are symmetric.*

(b) Take the isotropic and deviatoric parts of $\underset{\sim}{\sigma} = 2\mu\underset{\sim}{\varepsilon} + \lambda(\text{tr}\underset{\sim}{\varepsilon})\underset{\sim}{I}$ *and compare with Eq.(23.63).*

Do all fourth-order tensors commute with isotropic fourth-order tensors? Nope.

For second-order tensors, $\underset{\sim}{\mathbf{A}} \bullet \underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{B}} \bullet \underset{\sim}{\mathbf{A}}$ if $\underset{\sim}{\mathbf{A}}$ is isotropic. Stated differently, isotropy of $\underset{\sim}{\mathbf{A}}$ is sufficient to assert that $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ commute. This begs the question: *does a similar assertion hold for fourth-order tensors?* Is isotropy of a fourth-order tensor $\underset{\approx}{\mathbf{A}}$ sufficient to assert that $\underset{\approx}{\mathbf{A}}:\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{B}}:\underset{\approx}{\mathbf{A}}$? The answer is no.

If $\underset{\approx}{\mathbf{A}}$ is isotropic, Eq. (23.44) ensures existence of scalar coefficients such that

$$\underset{\approx}{\mathbf{A}} = \alpha(\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}) + \beta(\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}})^{\mathrm{L}} + \gamma(\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}})^{\Gamma} \tag{23.64}$$

so that, for any tensor $\underset{\sim}{\mathbf{X}}$,

$$\underset{\approx}{\mathbf{A}}:\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{X}} = \alpha(\mathrm{tr}\underset{\sim}{\mathbf{X}})(\underset{\sim}{\mathbf{I}}:\underset{\approx}{\mathbf{B}}) + \beta(\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{X}}) + \gamma(\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{X}})^T \tag{23.65a}$$

$$\underset{\approx}{\mathbf{B}}:\underset{\approx}{\mathbf{A}}:\underset{\sim}{\mathbf{X}} = \alpha(\mathrm{tr}\underset{\sim}{\mathbf{X}})(\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{I}}) + \beta\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{X}} + \gamma\underset{\approx}{\mathbf{B}}:\underset{\sim}{\mathbf{X}}^T \tag{23.65b}$$

Subtracting these two equations and asserting that the result must be zero for all $\underset{\sim}{\mathbf{X}}$ produces the following necessary and sufficient condition for the isotropic tensor $\underset{\approx}{\mathbf{A}}$ to commute with the general fourth-order tensor $\underset{\approx}{\mathbf{B}}$:

$$\alpha(B_{kkij} - B_{ijkk}) = 0 \tag{23.66a}$$

$$\gamma(B_{jipq} - B_{ijqp}) = 0 \tag{23.66b}$$

These constraints are met if $\underset{\approx}{\mathbf{B}}$ is both major and minor symmetric.

In the context of constitutive modeling, the question of commutativity with an isotropic fourth-order tensor typically arises in cases where $\gamma = 0$. In that case, a sufficient condition for commutativity is for $\underset{\approx}{\mathbf{B}}$ to be simply major-symmetric (with no requirement of minor symmetry). More generally, if $\gamma=0$, then commutativity is assured if

$$B_{3322} - B_{2233} = B_{1133} - B_{3311} = B_{2211} - B_{1122} \tag{23.67}$$

This condition is met if $\underset{\approx}{\mathbf{B}}$ is merely major-symmetric (with no requirement of minor symmetry). In damage mechanics, a fourth-order "**stiffness degradation**" tensor might be defined

$$\underset{\approx}{\boldsymbol{\psi}} = \underset{\approx}{\mathbf{C}}_0:\underset{\approx}{\mathbf{S}} \tag{23.68}$$

where $\underset{\approx}{\mathbf{C}}_0$ is the initial elastic stiffness tensor and $\underset{\approx}{\mathbf{S}} = \underset{\approx}{\mathbf{C}}^{-1}$ is the damaged compliance tensor, equal to the inverse of the damaged stiffness $\underset{\approx}{\mathbf{C}}$.

Incidentally, the stiffness degradation tensor is useful in simple Tsai-Wu yield (or failure) models for which the damaged yield function $f[\underset{\sim}{\boldsymbol{\sigma}}]$ is related to the initial yield function f_0 by

$$f[\underset{\sim}{\boldsymbol{\sigma}}] = f_0\left[\underset{\approx}{\boldsymbol{\psi}}:(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{b}})\right] \tag{23.69}$$

in which $\underset{\sim}{\mathbf{b}}$ is the backstress tensor (used to redefine the origin for elasticity for the Bauschinger effect).

If the initial stiffness is isotropic, the results of this section demonstrate that

$$\underset{\approx}{\boldsymbol{\psi}} = \underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{C}}_0 \tag{23.70}$$

In this case, $\underset{\approx}{\mathbf{C}}_0$ and $\underset{\approx}{\mathbf{S}}$ share a common set of eigentensors. To connect with the notion of damage that is understood by non-constitutive modelers,* we suggest defining a "**damage tensor**" as

$$\underset{\approx}{\mathbf{d}} = \underset{\approx}{\mathbf{P}}^{\text{sym}} - \underset{\approx}{\boldsymbol{\psi}}^{-1} \tag{23.71}$$

This tensor's nonzero eigenvalues $d_1, \ldots, d_6$ represent conventional damages, with associated eigentensors representing degradation of material response to specific deformation modes. An overall scalar measure of damage is

$$d = \left(\frac{1}{6}\underset{\approx}{\mathbf{d}}:\underset{\approx}{\mathbf{d}}\right)^{1/2} = \sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_6^2}{6}} \tag{23.72}$$

The factor of $\frac{1}{6}$ ensures that damage goes to unity as $\underset{\approx}{\mathbf{d}}$ approaches $\underset{\approx}{\mathbf{P}}^{\text{sym}}$.

The isotropic part of a fourth-order tensor

Fourth-order tensors are of type V_3^4, but they are also of type V_{81}^1. In other words, they may be regarded as 81-dimensional vectors. The set of isotropic fourth-order engineering tensors (IFOET) is closed under tensor addition and scalar multiplication. This means that any linear combination of IFOET tensors will itself be IFOET. Therefore, the set of all IFOET tensors forms a subspace of general fourth-order engineering tensor space. A general non-IFOET tensor can be therefore broken into its IFOET part plus "whatever is left over."

In the previous section, we showed that any IFOET tensor can be written as a linear combination of P_{ijkl}^{iso}, P_{ijkl}^{symdev}, and P_{ijkl}^{skew}. These three tensors therefore form a basis for the set of all IFOET tensors, and the IFOET subspace must be 3-dimensional. This basis is orthogonal (e.g., $P_{ijkl}^{\text{iso}} P_{ijkl}^{\text{skew}} = 0$), but it is not normalized. We can define an orthonormal basis for IFOET tensors as

$$\hat{P}_{ijkl}^{\text{iso}} \equiv P_{ijkl}^{\text{iso}} \tag{23.73a}$$

$$\hat{P}_{ijkl}^{\text{symdev}} \equiv \frac{P_{ijkl}^{\text{symdev}}}{\sqrt{5}} \tag{23.73b}$$

* Most people think of damage d as appearing in the multiplier of initial elastic stiffness E_0 such that the current stiffness is $E = (1-d)E_0$. Solving for d gives $d = 1 - \psi^{-1}$, where $\psi = E_0 E^{-1}$. Equation (23.71) simply upgrades this result to a tensor context, where the fourth-order tensor $\underset{\approx}{\mathbf{P}}^{\text{sym}}$ is the identity on the space of symmetric tensors.

$$\hat{P}^{\text{skew}}_{ijkl} \equiv \frac{P^{\text{skew}}_{ijkl}}{\sqrt{3}} . \tag{23.73c}$$

The denominators in this equation are the magnitudes of the tensors in the numerators, obtained by taking the square root of the fourth-order inner product of the tensors with themselves. The denominators can be easily remembered because the magnitude of any projector is the square root of the dimension of its range space. When P^{iso}_{ijkl}, P^{symdev}_{ijkl}, or P^{skew}_{ijkl} acts on an arbitrary second-order tensor, the result is, respectively, the isotropic, symmetric-deviatoric, or skew part of that tensor. These tensors have, respectively, 1, 5, and 3 independent components. Therefore, the Euclidean magnitudes of P^{iso}_{ijkl}, P^{symdev}_{ijkl}, and P^{skew}_{ijkl} are, respectively, $\sqrt{1}$, $\sqrt{5}$, and $\sqrt{3}$.

Whenever three m-dimensional *orthonormal* vectors $\underline{\boldsymbol{n}}_1$, $\underline{\boldsymbol{n}}_2$ and $\underline{\boldsymbol{n}}_3$, form a basis for a 3D linear subspace embedded in a larger m-dimensional vector space (type V^1_m), then any vector $\underline{\boldsymbol{x}}$ in the higher dimensional space can be projected to the 3D subspace by applying the operation

$$\underline{\boldsymbol{x}}^{\text{projected}} = \underline{\underline{\boldsymbol{P}}} * \underline{\boldsymbol{x}} , \tag{23.74}$$

where

$$\underline{\underline{\boldsymbol{P}}} = \underline{\boldsymbol{n}}_1\underline{\boldsymbol{n}}_1 + \underline{\boldsymbol{n}}_2\underline{\boldsymbol{n}}_2 + \underline{\boldsymbol{n}}_3\underline{\boldsymbol{n}}_3 , \tag{23.75}$$

and the "*" denotes the inner product in the V^1_m space.

We are interested in finding the IFOET part of a general fourth-order engineering tensor X_{ijkl}. This is accomplished by projecting that tensor to the IFOET subspace. Using Eq. (23.74), this operation is found by

$$X^{\text{IFOET}}_{ijkl} = P_{ijklpqrs} X_{pqrs} , \tag{23.76}$$

where the components of the eighth-order (V^8_3) tensor are found by using Eq. (23.73) as the orthonormal basis in Eq. (23.75). Namely,

DON'T WORRY ABOUT THIS ALPHABET SOUP -- IT SOON GETS EASIER.

$$P_{ijklpqrs} = P^{\text{iso}}_{ijkl} P^{\text{iso}}_{pqrs} + \frac{1}{5} P^{\text{symdev}}_{ijkl} P^{\text{symdev}}_{pqrs} + \frac{1}{3} P^{\text{skew}}_{ijkl} P^{\text{skew}}_{pqrs} . \tag{23.77}$$

Because most of the components of the isotropy basis are zero, you won't work directly with eighth-order tensors. The above equations are *operational*, intended to lend insight rather than computational usefulness. These equations are similar to the second-order operation $\underset{\approx}{\boldsymbol{I}} : \underset{\approx}{\boldsymbol{B}}$, which you would *not* use in practice because most of the components of

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the identity are zero. Instead, you would recognize that $\underset{\approx}{\boldsymbol{I}}:\underset{\approx}{\boldsymbol{B}}$ simply equals $\mathrm{tr}\underset{\approx}{\boldsymbol{B}}$ and be finished. Likewise, instead of using the above equations directly, fourth-order tensors are usually written as 9×9 matrices (or 6×6 if minor symmetric). Given the 9×9 Mandel matrix for a general tensor, its IFOET part is found as follows:

STEP 1. Set α equal to the sum of all components in the upper-left 3×3 submatrix.
Set β equal to the sum of the first six diagonal components
Set γ equal to the sum of the last three diagonal components

STEP 2. Set $a = \dfrac{(2\alpha - \beta)}{15}, \qquad b = \dfrac{(3\beta - \alpha)}{15}, \qquad c = \dfrac{\gamma}{3}$

STEP 3. The 9×9 Mandel matrix can be written as a 3×3 array of 3×3 matrices. Doing so, the IFOET part is

$$a\begin{bmatrix} [U] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} [I] & 0 & 0 \\ 0 & [I] & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [I] \end{bmatrix},$$

$$\text{where} \quad [U] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

STEP 4. The bulk modulus K and the shear modulus G (defined such that $3K$ is the coefficient of P^{iso}_{ijkl}, and $2G$ is the coefficient of $P^{\mathrm{symdev}}_{ijkl}$) are given by

$$K = \alpha/9 \qquad \text{and} \qquad G = \frac{b}{2} = \frac{(3\beta - \alpha)}{30}$$

A scalar measure of fourth-order isotropy (or anisotropy).

In much the same way that a scalar measure of isotropy was defined for second-order tensors in Eq. (23.16) or (23.17), the extent of isotropy of a fourth-order tensor may be quantified by comparing the magnitude of the tensor with the magnitude of its IFOET part. When we quantified isotropy of *second*-order tensors, we found the angle formed with the *one-dimensional* manifold of isotropic tensors. For fourth-order tensors the manifold of isotropic tensors is *three-dimensional* because it has three basis tensors (i.e., those given in Eq. 23.73). Thus, in this case, a scalar measure of isotropy can be defined via the geometrical analog of the angle between a vector and a plane. Namely, a scalar measure of isotropy for a fourth-order tensor can be defined analogously to Eq. (23.17) by

$$1 - \frac{2}{\pi}\mathrm{ArcCos}\left(\frac{\left\|\underset{\approx}{\boldsymbol{X}}^{\mathrm{IFOET}}\right\|}{\left\|\underset{\approx}{\boldsymbol{X}}\right\|}\right) \qquad \text{scalar measure of FOURTH-order tensor isotropy} \qquad (23.78)$$

This number ranges from 0 when the tensor has no isotropic part to 1 when it is purely isotropic. This sort of scalar measure can be very convenient when assessing laboratory measurements of elastic stiffnesses for materials modeling. Experimental error and slight material imperfections will always result in an anisotropic stiffness; however, an assumption of isotropy can be justified if Eq. (23.78) is nearly equal to 1. If so, the anisotropic part may be removed by applying Eq. (23.76). The above measure of tensor isotropy (or anisotropy if only the last term is used) has the disadvantage that it is not invariant under inversion (i.e., a different value results when the inverse of the tensor is substituted into the same formula). However, a similar

In terms of the parameters defined in the earlier algorithm for finding the IFOET part of a fourth-order tensor,

$$\left\|\underset{\approx}{\mathbf{X}}^{\text{IFOET}}\right\| = \sqrt{(3K)^2+5(2G)^2+3\gamma^2}$$
$$= \sqrt{3(3a^2+2ab+2b^2+9c^2)}$$
$$= \sqrt{\frac{1}{15}(2\alpha^2-2\alpha\beta+3\beta^2+5\gamma^2)} \tag{23.79}$$

What is *material* isotropy?

This section requires tensor calculus, so skip it if you are new to tensor math. It is kept here to help emphasize the distinction between isotropic tensors and isotropic material.

Material isotropy is quite different from tensor isotropy. Consider, as an example, classic hyperelasticity. Under this theory, the strain is derivable from an energy potential function such that*

$$\underset{\approx}{\varepsilon} = \frac{d\phi}{d\underset{\approx}{\sigma}}, \quad \text{or} \quad \varepsilon_{ij} = \frac{\partial\phi}{\partial\sigma_{ij}}. \tag{23.80}$$

The tangent elastic compliance tensor (inverse of stiffness) is defined

$$H_{ijkl} = \frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}, \quad \text{or, equivalently,} \quad H_{ijkl} = \frac{\partial^2\phi}{\partial\sigma_{ij}\partial\sigma_{kl}}. \tag{23.81}$$

If the potential function exists, this shows that the compliance will be major symmetric:

$$H_{ijkl} = H_{klij} \tag{23.82}$$

Conversely, given a model for strain as a function of stress, the energy potential exists (i.e., Eq. 23.80 can be integrated) if the compliance is major symmetric.

* Is no energy potential exists, then the material is called **hypoelastic**. The integrability test that determines whether or not a potential exists for a nonlinear compliance H_{ijkl} that depends on stress is $\partial H_{ijkl}/\partial\sigma_{mn} = \partial H_{ijmn}/\partial\sigma_{kl}$. If this condition is not satisfied, the model is hypoelastic.

Reb: insert a self-study exercise to demonstrate that the potential function exists for the ramberg osgood model. (get the tensor formulation from the Computational Inelasticity class notes.)

The material is said to possess **material isotropy** if the potential function evaluates to the same number regardless of the principal directions of stress. In other words, the material isotropy means that the potential function depends only on the stress *invariants*. Hence, the most general potential function for an elastically isotropic material is

$$\phi = \phi(I_1, J_2, J_3)\,. \tag{23.83}$$

Applying the chain rule* and Eqs. (20.128), the strain defined in Eq. (23.80) is therefore

$$\varepsilon_{ij} = \frac{\partial\phi}{\partial I_1}\delta_{ij} + \frac{\partial\phi}{\partial J_2}S_{ij} + \frac{\partial\phi}{\partial J_3}T_{ij}\,. \tag{23.84}$$

Now applying Eq. (23.81), and using the chain rule yet again to evaluate the second-order partial derivatives of the potential function, the fourth-order compliance is (after considerable simplification) found to be

Most general tangent compliance of an isotropic material

$$\boxed{\begin{aligned} \underset{\approx}{\mathbf{H}} = {} & 3\phi_{11}\underset{\approx}{\mathbf{P}}^{\text{iso}} + \phi_2\underset{\approx}{\mathbf{P}}^{sd} \\ & + \left(\phi_{12} - \tfrac{2}{3}\phi_3\right)(\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{S}} + \underset{\approx}{\mathbf{S}}\underset{\approx}{\mathbf{I}}) + \phi_3(\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{S}} + \underset{\approx}{\mathbf{S}}\underset{\approx}{\mathbf{I}})^{\lambda} \\ & + \phi_{22}\underset{\approx}{\mathbf{S}}\underset{\approx}{\mathbf{S}} + \phi_{31}(\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{T}} + \underset{\approx}{\mathbf{T}}\underset{\approx}{\mathbf{I}}) \\ & + \phi_{23}(\underset{\approx}{\mathbf{S}}\underset{\approx}{\mathbf{T}} + \underset{\approx}{\mathbf{T}}\underset{\approx}{\mathbf{S}}) + \phi_{33}\underset{\approx}{\mathbf{T}}\underset{\approx}{\mathbf{T}} \end{aligned}} \tag{23.85}$$

Here, subscripts on ϕ denote differentiation (ϕ_k is the partial with respect to the k^{th} invariant; $\phi_{\alpha\beta}$ similarly denotes the second partial derivative). The superscript operator "λ" denotes minor-symmetrization of the leaf operator "L" [see Eq. (17.79) on page 476.]

* If you are a newcomer to tensor analysis and therefore not yet familiar with how to differentiate tensor functions using the chain rule, you might want to skip this section for now.

Note that H_{ijkl} has anisotropic terms even though the material is isotropic! This observation provides a key reason to be skeptical of material models that employ an isotropic elastic stiffness together with a pressure-dependent shear modulus (see part "b" of Exercise 23.1) *Reb: numbering issue: I have two Exercise numbers the same Cross-reference is showing more than one paragraph with the same number.*

Exercise 23.1 This question explores conditions for an isotropic material to have isotropic compliance.
(a) Show that the compliance is isotropic in the limit of ZERO stress (i.e., to zero-order accuracy). In this case, find expressions for the shear and bulk modulus in terms of derivatives of the potential function.
(b) For nonzero stress, prove that an isotropic material has an isotropic tangent compliance only if the shear modulus is constant and the bulk modulus depends, at most, on pressure.
(c) Find the most general form of the energy potential for which the tangent compliance is independent of stress.
(d) Find the most general form of the energy potential for which the tangent compliance varies linearly with stress.

Hint: (a) for zero stress, which terms in Eq. (23.85) fall out? Compare the result with the definitions of shear and bulk modulus in Eq. (23.52). (b) set the coefficients of the anisotropic terms to zero. Doing that gives you a set of PDEs that lead to the stated conclusion. (c) use the result from part "b" to infer that dependence of the bulk modulus on the first stress invariant must be quadratic, while dependence on the second invariant must be linear, ultimately giving $\phi = \frac{J_2}{2G} + \frac{I_1^2}{18K} + \varepsilon_v^0 I_1$, *where K and G are constants (equal to the initial shear and bulk moduli), and* ε_v^0 *is a constant (equal to the initial volumetric strain). If the initial strain is zero, this shows that two experiments (measuring shear and bulk modulus) are required to characterize the stress-strain response to first order accuracy (d) In the compliance formula, set coefficients of the terms* $O(\text{stress}^3)$ *and* $O(\text{stress}^2)$ *equal to zero to obtain PDEs solvable for the general form of the potential function. The final result is* $\phi = \frac{J_2}{2G} + \frac{I_1^2}{18K} + \varepsilon_v^0 I_1 + \alpha I_1^3 + \beta I_1 J_2 + \gamma J_3$, *where* $\{\alpha, \beta, \gamma\}$ *are constants. This shows that a second-order accurate description of the stress-strain response of an isotropic material requires three more material parameters than first-order accuracy.*

In general, the tangent compliance is isotropic only in the limit of zero stresses. The most general form of nonlinear elasticity for which the elastic tangent tensor will *also* be isotropic is one in which the bulk modulus depends purely on I_1, while the shear modulus is *constant*. Constitutive modelers often employ a non-constant shear modulus in combination with an isotropic tangent stiffness, which (as shown in part "b" of Exercise 23.1) carries with it an implicit consequence that their elastic model is not derivable from a potential. Such a result contradicts thermodynamics, but it can be shown [*cite Fuller-Brannon*] that the error is small if the strength of the material is not strongly pressure dependent. By applying a stress to a material, you will introduce **deformation-induced anisotropy**. The fact that an isotropic material generally has an anisotropic stiffness should make a lot of sense if you imagine what happens when a kitchen sponge (initially in the shape of a sphere) is compressed. The initially spherical pores become ellipsoids in compression, thus introducing material texture (i.e., anisotropy) under loading. The material is still elastic because the texture disappears upon release of the load. The material is still isotropic because the induced anisotropy is unchanged if the sponge is first reoriented before applying the load.

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TENSOR ALGEBRA

Material isotropy for hyperelasticity in large deformations.

This section refers to continuum kinematics tensors (such as the deformation gradient) as well as stress definitions (such as the first Piola-Kirchhoff stress) that are defined in Chapter 38 on page 1070. Accordingly, this section should be regarded as an advanced topic to be skipped by newcomers to tensor analysis and continuum mechanics.

The preceding section referred generically to stress and strain, and that section worked with stress being the independent variable from which strain was determined. In large deformation continuum mechanics, there are multiple definitions of stress and strain. This section revisits the same topics from the perspective that is more commonly used to develop large-deformation hyperelasticity models.

A thermoelastic material is one for which the internal energy depends only on the deformation gradient tensor $\underset{\approx}{\boldsymbol{F}}$ and the entropy. Thus, if entropy is held constant (which, for a reversible material is tantamount to loading in adiabatic conditions), the internal energy depends only on $\underset{\approx}{\boldsymbol{F}}$. Through a Legendre transformation, thermodynamics considerations show that the Helmholtz free energy of a thermoelastic material depends only on $\underset{\approx}{\boldsymbol{F}}$ and the temperature. Thus, if temperature is held constant (which is typical of slow loading tests in a laboratory), the free energy depends only on $\underset{\approx}{\boldsymbol{F}}$. Adiabatic and isothermal conditions are just two examples of situations for which thermal variables are controlled sufficiently to regard energy to depend only on $\underset{\approx}{\boldsymbol{F}}$. In this case, there exists a potential function W, having units of energy per reference volume, such that

$$\underset{\approx}{\boldsymbol{P}} = \frac{dW}{d\underset{\approx}{\boldsymbol{F}}} \tag{23.86}$$

where, from the discipline of Continuum Mechanics, $\underset{\approx}{\boldsymbol{P}}$ is the first Piola-Kirchhoff stress, defined in terms of the Cauchy stress $\underset{\approx}{\boldsymbol{\sigma}}$ by

$$\underset{\approx}{\boldsymbol{P}} = \underset{\approx}{\boldsymbol{\sigma}} \bullet \underset{\approx}{\boldsymbol{F}}^C = J\underset{\approx}{\boldsymbol{\sigma}} \bullet \underset{\approx}{\boldsymbol{F}}^{-T} \tag{23.87}$$

where the superscript "C" is the cofactor operation and $J = \det\underset{\approx}{\boldsymbol{F}}$ is the Jacobian of the deformation. By using the principle of material frame indifference (PMFI; see page 1154), it can be shown that

$$\underset{\approx}{\boldsymbol{S}} = \frac{dW}{d\underset{\approx}{\boldsymbol{E}}}, \tag{23.88}$$

where

$$\underset{\approx}{\boldsymbol{S}} = \underset{\approx}{\boldsymbol{F}}^{-1} \bullet (J\underset{\approx}{\boldsymbol{\sigma}}) \bullet \underset{\approx}{\boldsymbol{F}}^{-T} = \text{“second Piola-Kirchhoff (PK2) stress”} \tag{23.89}$$

and

$$\underset{\approx}{\boldsymbol{E}} = \tfrac{1}{2}(\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}}) = \text{“Green-Lagrange (or just Lagrange) strain”} \tag{23.90}$$

The Green-Lagrange strain is often written as

$$\underset{\approx}{\boldsymbol{E}} = \tfrac{1}{2}(\underset{\approx}{\boldsymbol{C}} - \underset{\approx}{\boldsymbol{I}}) \tag{23.91}$$

where

$$\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{F}}^T \bullet \underset{\approx}{\mathbf{F}} = \text{“right Cauchy-Green tensor”} \tag{23.92}$$

Then Eq. (23.88) becomes

$$\underset{\approx}{\mathbf{S}} = 2\frac{dW}{d\underset{\approx}{\mathbf{C}}}, \tag{23.93}$$

For an isotropic material, the potential W is taken to depend only on the invariants of $\underset{\approx}{\mathbf{C}}$,*

$$I_1 = \mathrm{tr}\underset{\approx}{\mathbf{C}}, \qquad \text{for which} \quad \frac{dI_1}{d\underset{\approx}{\mathbf{C}}} = \underset{\approx}{\mathbf{I}} \tag{23.94a}$$

$$I_2 = \mathrm{tr}\underset{\approx}{\mathbf{C}}^C = \frac{1}{2}(I_1^2 - \mathrm{tr}(\underset{\approx}{\mathbf{C}}^2)) \qquad \text{for which} \quad \frac{dI_2}{d\underset{\approx}{\mathbf{C}}} = I_1\underset{\approx}{\mathbf{I}} - \underset{\approx}{\mathbf{C}} \tag{23.94b}$$

$$I_3 = \det\underset{\approx}{\mathbf{C}} \qquad \text{for which} \quad \frac{dI_3}{d\underset{\approx}{\mathbf{C}}} = \underset{\approx}{\mathbf{C}}^C = I_3\underset{\approx}{\mathbf{C}}^{-1} \tag{23.94c}$$

The derivatives are taken from Eq. (20.15) on page 510, simplified using symmetry of $\underset{\approx}{\mathbf{C}}$.

For an isotropic material, the potential function depends only on these invariants. Therefore, applying the chain rule to Eq. (23.86) gives

$$\tfrac{1}{2}\underset{\approx}{\mathbf{S}} = \frac{dW}{d\underset{\approx}{\mathbf{C}}} = \frac{\partial W}{\partial I_1}\frac{dI_1}{d\underset{\approx}{\mathbf{C}}} + \frac{\partial W}{\partial I_2}\frac{dI_2}{d\underset{\approx}{\mathbf{C}}} + \frac{\partial W}{\partial I_3}\frac{dI_3}{d\underset{\approx}{\mathbf{C}}} \tag{23.95}$$

As discussed on pages 15 and 772, note that ∂ is used in some derivatives, while “d” is used in others. Defining $W_k = \partial W/\partial I_k$ and using the derivatives in Eq. (23.94),

$$\tfrac{1}{2}\underset{\approx}{\mathbf{S}} = W_1\underset{\approx}{\mathbf{I}} + W_2(I_1\underset{\approx}{\mathbf{I}} - \underset{\approx}{\mathbf{C}}) + W_3 I_3\underset{\approx}{\mathbf{C}}^{-1} \tag{23.96}$$

Grouping terms by increasing powers on $\underset{\approx}{\mathbf{C}}$ gives

$$\boxed{\tfrac{1}{2}\underset{\approx}{\mathbf{S}} = W_3 I_3\underset{\approx}{\mathbf{C}}^{-1} + (W_1 + W_2 I_1)\underset{\approx}{\mathbf{I}} - W_2\underset{\approx}{\mathbf{C}}} \tag{23.97}$$

Using Eq. (23.89), the Cauchy stress is then

$$\boxed{\underset{\approx}{\boldsymbol{\sigma}} = \frac{2}{J}(W_3 I_3\underset{\approx}{\mathbf{I}} + (W_1 + W_2 I_1)\underset{\approx}{\mathbf{B}} - W_2\underset{\approx}{\mathbf{B}}^2)} \tag{23.98}$$

where

$$\underset{\approx}{\mathbf{B}} = \underset{\approx}{\mathbf{F}} \bullet \underset{\approx}{\mathbf{F}}^T = \text{“left Cauchy-Green tensor”} \tag{23.99}$$

The PK2 stress is a nonlinear function of the Green-Lagrange strain, which implies that the *rate* of the PK2 stress is linear with respect to the rate of Green-Lagrange strain:

* Here, we have elected to use the characteristic invariants of $\underset{\approx}{\mathbf{C}}$. An alternative invariant triplet, based on the isotropic-isochoric (dilation-distortion) decomposition of $\underset{\approx}{\mathbf{F}}$, is discussed later on page 637.

$$\dot{\mathbf{S}} = \mathbf{T}:\dot{\mathbf{E}}, \text{ where } \mathbf{T} = \frac{d\mathbf{S}}{d\mathbf{E}} = \frac{d^2 W}{d\mathbf{E}d\mathbf{E}} = 4\frac{d^2 W}{d\mathbf{C}d\mathbf{C}} = 2\frac{d\mathbf{S}}{d\mathbf{C}} \tag{23.100}$$

Keep in mind: the stress rate is linear with respect to strain rate, which means that the tangent stiffness $\mathbf{T}$ is independent of the strain rate. For nonlinear elasticity, the tangent stiffness is still nonlinearly dependent on the strain itself.

Using Eq. (23.97),

$$\begin{aligned} \frac{1}{4}\mathbf{T} = {} & \mathbf{C}^{-1}\frac{d(W_3 I_3)}{d\mathbf{C}} + \mathbf{I}\frac{d(W_1 + W_2 I_1)}{d\mathbf{C}} - \mathbf{C}\frac{dW_2}{d\mathbf{C}} \\ & + W_3 I_3 \frac{d(\mathbf{C}^{-1})}{d\mathbf{C}} - W_2\frac{d\mathbf{C}}{d\mathbf{C}} \end{aligned} \tag{23.101}$$

The chain rule may be used to evaluate the derivatives in the first line, as was done to obtain Eq. (23.95). The last derivative is *nominally* the fourth-order identity tensor, but (as explained later in Eq. 27.370) the fourth-order identity *in symmetric tensor space* is required. Specifically,

$$\frac{d\mathbf{C}}{d\mathbf{C}} = \mathbf{P}^{\text{sym}} = \mathbf{I}\wedge\mathbf{I} \tag{23.102}$$

where $\mathbf{P}^{\text{sym}}$ is the symmetry projection defined in Eq. (14.14) on page 382, and the "^" operator is defined in Eq. (17.58) on page 474. Similarly, using Eq. (14.44) on page 386,

$$\frac{d(\mathbf{C}^{-1})}{d\mathbf{C}} = -\mathbf{C}^{-1}\wedge\mathbf{C}^{-1} \tag{23.103}$$

Following through with the derivatives and collecting terms gives

$$\begin{aligned} \frac{1}{4}\mathbf{T} = {} & (W_{33}I_3^2 + W_3 I_3)\mathbf{C}^{-1}\mathbf{C}^{-1} + (W_2 + W_{11} + 2W_{12}I_1 + W_{22}I_1^2)\mathbf{I}\mathbf{I} + W_{22}\mathbf{C}\mathbf{C} \\ & + 2[(W_{13}I_3 + W_{23}I_3 I_1)\mathbf{C}^{-1}\mathbf{I} - W_{23}I_3\mathbf{C}^{-1}\mathbf{C} - (W_{12} + W_{22}I_1)\mathbf{C}\mathbf{I}]^M \\ & + W_3 I_3 \mathbf{C}^{-1}\wedge\mathbf{C}^{-1} - W_2\mathbf{I}\wedge\mathbf{I}, \end{aligned} \tag{23.104}$$

where W_{ij} denotes second partial derivatives (similar to W_k). All terms in the first row involve dyadic multiplication. The terms in the first line are all dyadic multiplications of tensors with themselves, thus giving major symmetry. The terms in the last middle line are major symmetric by virtue of the "M" operator, defined on page 476, applied to all terms in that line. Major symmetry of the final line results from the symmetry of $\mathbf{C}$ and $\mathbf{I}$.

Some researchers transform this PK2 stiffness with a "push forward" Rayleigh product, defined on page 1022, to obtain

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$$\frac{1}{4}\underset{\approx}{\mathbf{F}}>\underset{\approx}{\mathbf{T}} = (W_{33}I_3^2 + W_3I_3)\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{I}} + (W_2 + W_{11} + 2W_{12}I_1 + W_{22}I_1^2)\underset{\approx}{\mathbf{B}}\underset{\approx}{\mathbf{B}} + W_{22}\underset{\approx}{\mathbf{B}}^2\underset{\approx}{\mathbf{B}}^2$$
$$+ 2[(W_{13}I_3 + W_{32}I_3I_1)\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{B}} - W_{23}I_3\underset{\approx}{\mathbf{I}}\underset{\approx}{\mathbf{B}}^2 - (W_{12} + W_{22}I_1)\underset{\approx}{\mathbf{B}}^2\underset{\approx}{\mathbf{B}}]^M$$
$$+ W_3I_3\underset{\approx}{\mathbf{I}}\hat{}\underset{\approx}{\mathbf{I}} - W_2\underset{\approx}{\mathbf{B}}\hat{}\underset{\approx}{\mathbf{B}} \tag{23.105}$$

Note that this result is identical to Eq. (23.105) except that $\underset{\approx}{\mathbf{C}}^{-1}$, $\underset{\approx}{\mathbf{I}}$, and $\underset{\approx}{\mathbf{C}}$ in Eq. (23.104) are replaced, respectively, with $\underset{\approx}{\mathbf{I}}$, $\underset{\approx}{\mathbf{B}}$, and $\underset{\approx}{\mathbf{B}}^2$ in Eq. (23.105). See Exercise 27.7 on page 802 for details.

Large deformation hyperelasticity using dilation-distortion invariants.

As proved on page 607, the deformation gradient may be decomposed into the form

$$\underset{\approx}{\mathbf{F}} = J^{1/3}\underset{\approx}{\bar{\mathbf{F}}}, \tag{23.106}$$

where

$$J = \det\underset{\approx}{\mathbf{F}}\text{, thus characterizing volume change} \tag{23.107}$$

$$\det\underset{\approx}{\bar{\mathbf{F}}} = 1\text{, thus characterizing shape change} \tag{23.108}$$

Defining

$$\underset{\approx}{\bar{\mathbf{C}}} = \underset{\approx}{\bar{\mathbf{F}}}^T \bullet \underset{\approx}{\bar{\mathbf{F}}} = J^{-2/3}\underset{\approx}{\mathbf{C}} \tag{23.109}$$

an alternative invariant triplet is

$$J = \det\underset{\approx}{\mathbf{F}} = I_3^2 \tag{23.110}$$

$$\bar{I}_1 = \mathrm{tr}\underset{\approx}{\bar{\mathbf{C}}} = J^{-2/3}I_1 \tag{23.111}$$

$$\bar{I}_2 = \mathrm{tr}\underset{\approx}{\bar{\mathbf{C}}}^C = \frac{1}{2}(\bar{I}_1{}^2 - \mathrm{tr}(\underset{\approx}{\bar{\mathbf{C}}}^2)) = \left(\frac{1}{2}\right)J^{-4/3}I_2 \tag{23.112}$$

Finish this, and check against the Wikipedia article.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z